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M. S. Longuet-Higgins

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MASS TRANSPORT IN WATER WAVES

By M. S. LONGUET-HIGGINS

Trinity College, University of Cambridge

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CONTENTS

It was shown by Stokes that in a water wave the particles of fluid possess, apart from their orbital motion, a steady second-order drift velocity (usually called the mass-transport velocity). Recent experiments, however, have indicated that the mass-transport velocity can be very different from that predicted by Stokes on the assumption of a perfect, non-viscous fluid. In this paper a general theory of mass transport is developed, which takes account of the viscosity, and leads to results in agreement with observation.

Part I deals especially with the interior of the fluid. It is shown that the nature of the motion in the interior depends upon the ratio of the wave amplitude a to the thickness δ of the boundary layer: when a^2/δ^2 is small the diffusion of vorticity takes place by viscous 'conduction'; when a^2/δ^2 is large, by convection with the mass-transport velocity. Appropriate field equations for the stream function of the mass transport are derived. The boundary layers, however, require separate consideration.

In part II special attention is given to the boundary layers, and a general theory is developed for two types of oscillating boundary: when the velocities are prescribed at the boundary, and when the stresses are prescribed. Whenever the motion is simple-harmonic the equations of motion can be integrated exactly. A general method is described for determining the mass transport throughout the fluid in the presence of an oscillating body, or with an oscillating stress at the boundary.

In part III, the general method of solution described in parts I and II is applied to the cases of a progressive and a standing wave in water of uniform depth. The solutions are markedly different from the perfect-fluid solutions with irrotational motion. The chief characteristic of the progressive-wave solution is a strong forward velocity near the bottom. The predicted maximum velocity near the bottom agrees well with that observed by Bagnold.

PART I. THE INTERIOR OF THE FLUID

1. Introduction

As was pointed out by Stokes in a classical memoir (1847), the individual particles in a progressive, irrotational wave do not describe exactly closed paths; besides their orbital motion they possess also a second-order mean velocity (called the mass-transport velocity) in the direction of wave propagation. If the equation of the free surface is

$$z = a\,\mathrm{e}^{\mathrm{i}(kx-\sigma t)} + O(a^2k), \tag{1}$$

where x and z are horizontal and vertical co-ordinates (z measured downwards), t is the time, a is the wave amplitude, $k = 2\pi \div$ wave-length, and $\sigma = 2\pi \div$ wave period, then Stokes's expression for the mass-transport velocity $\overline{U}$ is equivalent to

$$\overline{U} = \frac{a^2\sigma k \cosh 2k(z-h)}{2\sinh^2 kh} + C, \tag{2}$$

where h is the depth and C is an arbitrary constant. If it is assumed that the total horizontal transport is zero, we must have

$$C = -\frac{a^2\sigma \sinh 2kh}{4h \sinh^2 kh} = -\frac{a^2\sigma}{2h}\coth kh. \tag{3}$$

In deep water ($kh \gg 1$) equation (2) becomes simply

$$\overline{U} = a^2\sigma k\,\mathrm{e}^{-2kz}. \tag{4}$$

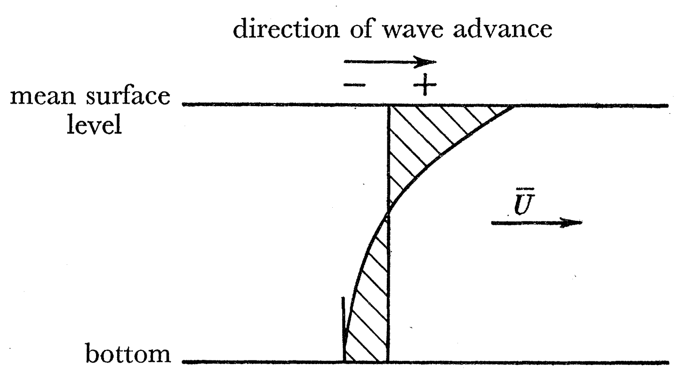

Figure 1. A typical profile of the mass-transport velocity in a progressive, irrotational wave ($kh = 1{\cdot}0$).

The velocity profile for a typical ratio of depth to wave-length ($kh = 1$) is shown in figure 1. It will be seen that the velocity increases steadily with height above the bottom and that on the bottom itself the velocity gradient is zero. Both these features can be shown to be necessary consequences of the irrotational character of the motion, and not to depend on the smallness of the wave amplitude as assumed by Stokes (an elegant geometrical proof for waves in deep water was given by Rayleigh (1876); proofs for finite depths have been given by Ursell (1953) and Longuet-Higgins (1953)).

The irrotational wave is not the only type of wave theoretically possible in a perfect fluid: in the exact solution of Gerstner (1809) and Rankine (1863) the particles describe exactly circular orbits, and the mass-transport velocity vanishes identically; indeed, Dubreil-Jacotin (1934) has shown that a wave motion may be superposed upon a steady stream having an arbitrary velocity distribution, so that the mass-transport velocity could take any desired value. The hypothesis of irrotational motion was assumed by Stokes on the ground that, under conservative forces, no vorticity can be generated in the interior of a uniform fluid, even with viscosity; if, therefore, the motion is started from rest it must initially be irrotational. The mass transport is then uniquely determined.

However, the mass-transport velocities observed in laboratory experiments may differ markedly from those predicted by the irrotational theory, especially in water of moderate depth. Thus Bagnold (1947) has found a strong forward velocity near the bottom and a weaker backward velocity at higher levels—the exact opposite of the Stokes velocity distribution. Other observers (Caligny 1878; King 1948) have found a forward drift both near the bottom and near the free surface, with a backward drift between.

It appears, therefore, that some assumption on which Stokes's theory is based is not valid. Now, in the theory of perfect fluids it is supposed that at a solid boundary the fluid may 'slip', i.e. that it may have a tangential velocity relative to the boundary. In fact, however, the particles of fluid in contact with the boundary must have the same velocity as the boundary itself; on the bottom, for example, they must be at rest. But quite near the bottom the fluid is observed to be in motion with velocities comparable to that in the interior of the fluid, so that in general there must be a strong velocity gradient near the bottom. This implies that there is in fact strong vorticity in the neighbourhood of the bottom; and it will be seen that, even if the vorticity is confined to a layer of infinitesimal thickness, the total amount of vorticity must still be finite. In an oscillating motion this vorticity will be of alternating sign; and the question then presents itself: will any of the vorticity spread into the interior of the fluid, or will it remain in the neighbourhood of the boundary?

In considering the diffusion of vorticity, the viscosity of the fluid must be taken into consideration; for, although the viscous terms in the equations of motion vanish when the motion is irrotational, they do not do so when there is vorticity. Although the viscosity may be small it cannot be neglected near the boundaries; for it is found that, as the viscosity tends to zero, so the thickness of the boundary layer decreases; the viscous terms in the equation of horizontal motion, which depend upon the second normal derivative of the velocity, remain of finite magnitude.

A straightforward method of taking into account the viscosity would be to proceed by successive approximations as in Stokes's solution for a perfect fluid; that is, in the first approximation to neglect all terms proportional to the square of the displacement; in the second approximation to neglect all terms proportional to the cube, and so on—all the viscous terms being retained. For surface waves in water of uniform depth there are now four boundary conditions: both components of velocity must vanish on the bottom, and both components of stress must vanish at the free surface. This was the method by which the present author originally approached the problem. The first approximation, which had been calculated by Hough (1896) and Bassett (1888), is practically identical with the perfect-fluid solution except that there are now transitional boundary layers at the bottom and at the free surface, and that the motion has a small attenuation, either with the horizontal co-ordinate x or with the time t (Hough and Basset considered only the latter case).

To obtain the mass transport, the present author took the solution to a second approximation; and when this was done some new and unexpected features appeared. These will be briefly described here, although another method, as will be explained below, was later found to be more satisfactory.

Two cases were considered: the progressive wave and the standing wave. On the assumption that the total mass transport in a horizontal direction was zero, a unique solution for the mass-transport distribution was found. But, when the viscosity was made to tend to zero, the limiting velocity distribution was different from the irrotational, perfect-fluid solution. The thickness of the boundary layers at the bottom and at the free surface tended to zero; but the mass-transport velocity just outside these layers tended to a value different from zero and from that in the Stokes solution. In the progressive wave, the forward velocity near the bottom (i.e. just beyond the boundary layer) was given by

$$\overline{U} = \frac{5}{4}\frac{a^2\sigma k}{\sinh^2 kh}, \tag{5}$$

and the velocity gradient near the surface was given by

$$\frac{\partial \overline{U}}{\partial z} = -4a^2\sigma k \coth kh; \tag{6}$$

this is twice the corresponding value for the irrotational wave (cf. equation (2)). In the interior of the fluid the velocity distribution was given by the sum of the distribution (2) and a parabolic distribution, which was adjusted so that equations (5) and (6) and the condition that the total horizontal flow should be zero were all satisfied. Some theoretical velocity profiles, for different ratios of depth to wave-length, will be illustrated in figure 6, part III.

The case of a standing wave, in which the surface elevation is given by

$$z = 2a \cos kx \cos \sigma t, \tag{7}$$

had already been partly evaluated by Rayleigh (1883), who showed that there must be a circulation in cells of width one-quarter of a wave-length, very similar to that occurring in a Kundt's tube. The magnitude of the circulation is independent of the viscosity, when this is small. Rayleigh considered only the case of deep water, and he did not take into account the boundary conditions at the free surface. In the general case when the depth is finite the present author found that the mass-transport velocity near the bottom (just outside the boundary layer) is given by

$$\overline{U} = -\frac{3}{2}\frac{a^2\sigma k}{\sinh^2 kh}\sin 2kx; \tag{8}$$

the velocity gradient near the free surface is zero,

$$\frac{\partial \overline{U}}{\partial z} = 0. \tag{9}$$

(The distribution of the mass transport in a typical standing wave will be illustrated in figure 7, part III.) The solution again differs from the corresponding solution when the motion is irrotational; in an irrotational standing wave the mass-transport velocity vanishes everywhere.

An interpretation of these results may be given as follows. Suppose the motion is generated from rest by conservative forces, or by propagation of the waves from outside into the region considered. Then at first the motion in the interior will be irrotational, and the mass transport will be given by Stokes's expression. But this state is not permanent; vorticity will diffuse inwards from the boundary layers at the bottom and at the free surface until a quasi-steady state, given by the viscous solution, is obtained. Thus the Stokes solution describes the initial motion (except very near the bottom); the viscous solution describes the final motion.

However, the method by which these results were derived is open to criticism: the process of approximation involves, in general, the neglect of the inertia terms in the equations of motion compared with the viscous terms; and this implies, as is shown in this part of the present paper, that the amplitude a of the motion should be small compared with the thickness δ of the boundary layer (δ is defined as $(2\nu/\sigma)^{\frac{1}{2}}$, where ν is the kinematic viscosity). It can also be shown that, unless $a \ll \delta$, it is not permissible to use Stokes's classical method of obtaining the boundary conditions at the free surface, for this involves expansion in a Taylor series, which is only valid if the displacement of the free surface is small compared with all other distances involved. Since the thickness of the boundary layer may be of the order of a few millimetres only, this condition seems to restrict the validity of the solution to very small waves indeed.

In this paper a different, and more general, approach is adopted. We start from the two fundamental assumptions that the velocity is periodic in time, and that the motion can be expressed as a perturbation of a state of rest. A general definition of the mass-transport velocity $\overline{\mathbf{U}}$ can then be given (see § 2), and equations of motion for $\overline{\mathbf{U}}$ can be derived. On examining these equations it is found that the expression for the diffusion of the vorticity consists of two parts. The first represents viscous diffusion, similar to the diffusion of heat in a solid, and the second represents diffusion by convection with the mass-transport velocity itself. These two sets of terms may be called 'conduction' and 'convection' terms respectively. The equations used by Stokes and Rayleigh are only valid, in the interior of the fluid, when the convection terms are small compared with the conduction terms, which restricts the solution to waves of very small amplitude ($a \ll \delta$). If, on the other hand, $a \gg \delta$, the motion is governed by convection; there is then a quite different field equation for the motion in the interior of the fluid (see § 4).

The boundaries, however, require special consideration, on account of the large velocity gradients encountered there. These are treated in part II, again in a general manner, so that the results could be applied to motions other than those of a standing or progressive wave in uniform depth. A general, oscillatory motion of the boundary is assumed, and moving co-ordinates relative to the boundary are taken. A boundary-layer approximation is made, similar to that used by Schlichting (1932) for a cylinder oscillating in an infinite fluid. Two different types of boundary layer are considered: first when the normal and tangential components of velocity at the boundary are prescribed (a special case being a fixed boundary or bottom); secondly, when the normal and tangential stresses are prescribed (a special case being a free surface, when both components of stress must vanish). In both cases the equations of motion can be integrated through the boundary layer, although the order of magnitude of the velocity gradients is different. In the first case the mass-transport velocity beyond the boundary layer (i.e. just in the interior of the fluid) is determined in terms of the boundary conditions and the known first-order motion; it differs in general from the mass-transport velocity at the boundary itself. In the second case it is the normal gradient of the mass-transport velocity which is determined just beyond the boundary layer, and this also differs from the velocity gradient at the surface itself.

The boundary-layer method just described has the advantage of not depending for its validity on the smallness of the ratio a/δ. By combining the new 'boundary conditions' with one or other of the field equations for the interior of the fluid which are derived in part I, the mass-transport velocity throughout the field can be completely evaluated. In part III the method is applied to the special cases of the progressive and standing waves in water of uniform depth. The 'conduction solution', i.e. the solution for small values of a/δ, is identical with that obtained by the method of successive approximations described above, as one would expect. The 'convection solution', however, is indeterminate for the progressive wave, and for the standing wave there are infinitely many solutions. Indeed, is seems very probable that for such large wave amplitudes the mass-transport velocity in the interior of the fluid is unstable; the assumption of periodicity then breaks down.

However, the solution in the boundary layers is still well determined, and is suitable for comparison with observation. In the last section of part III the experiments of Bagnold (1947) and others are discussed, and rather good agreement with the theory is found.

2. Definition of the mass-transport velocity

When the motion is not progressive, an exact definition of the mass-transport velocity for waves of finite amplitude, such as was given by Rayleigh (1876), is no longer possible; but for small motions a definition may be given as follows.

Let $\mathbf{u}(\mathbf{x}, t)$ denote the velocity at the point $\mathbf{x}$, $= (x, y, z)$, at time t. We assume, first, that the motion is periodic in time with period τ:

$$\mathbf{u}(\mathbf{x}, t+\tau) = \mathbf{u}(\mathbf{x}, t); \tag{10}$$

secondly, that $\mathbf{u}$ is expressible asymptotically as a power series:

$$\mathbf{u} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \ldots, \tag{11}$$

where ϵ is a small quantity and $\mathbf{u}_1$, $\mathbf{u}_2$, etc., are of order l/τ. Here l denotes a typical length in the geometry of the system, for example, the wave-length if the motion is periodic in space. Equation (11) implies that we are considering the motion as a perturbation of a state of rest. The order of magnitude of the displacements is ϵl, or a, where a denotes the wave amplitude, so that ϵ is of order a/l. Thirdly, if a bar denotes the mean value with respect to time over a complete period, we assume

$$\overline{\mathbf{u}_1} = 0, \tag{12}$$

that is, there are no steady first-order currents. It may not, however, be assumed that $\overline{\mathbf{u}_2}$ is zero.

Let $\mathbf{U}(\mathbf{x}_0, t)$ denote the velocity of the particle whose co-ordinates at time $t = 0$ are $\mathbf{x}_0$. Then the displacement of the particle from its original position is

$$\int_0^t \mathbf{U}\,\mathrm{d}t. \tag{13}$$

We have therefore

$$\mathbf{U} = \mathbf{u}\left(\mathbf{x}_0 + \int_0^t \mathbf{U}\,\mathrm{d}t, t\right), \tag{14}$$

$$= \mathbf{u}(\mathbf{x}_0, t) + \int_0^t \mathbf{U}\,\mathrm{d}t \cdot \operatorname{grad} \mathbf{u}(\mathbf{x}_0, t) + \dots, \tag{15}$$

by Taylor's theorem. Since $\mathbf{U}$ is of a same order as $\mathbf{u}$ we assume that

$$\mathbf{U} = \epsilon \mathbf{U}_1 + \epsilon^2 \mathbf{U}_2 + \dots, \tag{16}$$

whence, on substituting in (15) and equating coefficients of ϵ and ϵ^2, we have

$$\mathbf{U}_1 = \mathbf{u}_1, \tag{17}$$

$$\mathbf{U}_2 = \mathbf{u}_2 + \int_0^t \mathbf{u}_1\,\mathrm{d}t \cdot \operatorname{grad} \mathbf{u}_1, \tag{18}$$

and therefore

$$\overline{\mathbf{U}_1} = \overline{\mathbf{u}_1} = 0, \tag{19}$$

$$\overline{\mathbf{U}_2} = \overline{\mathbf{u}_2} + \overline{\int^t \mathbf{u}_1\,\mathrm{d}t \cdot \operatorname{grad} \mathbf{u}_1}. \tag{20}$$

The lower limit of integration in (20) has been omitted, since it contributes nothing to the mean value. Thus, besides the first-order oscillatory velocity $\epsilon \mathbf{U}_1$, each particle possesses a steady drift velocity given by

$$\overline{\mathbf{U}} = \epsilon^2 \overline{\mathbf{U}_2} = \epsilon^2\left(\overline{\mathbf{u}_2} + \overline{\int \mathbf{u}_1\,\mathrm{d}t \cdot \operatorname{grad} \mathbf{u}_1}\right) \tag{21}$$

to the second order of approximation. If $\mathbf{U}_3$, $\mathbf{U}_4$, etc., are calculated, they are found to be aperiodic in general, so that no mean value independent of the initial value of t can be assigned to them. Indeed, $\mathbf{U}$ cannot in general be expected to be a periodic function of t, since in the course of time a particle may drift into a region where the motion is quite different from that at its initial position. The progressive wave is an exception, since each particle remains at a nearly constant level; but the period of the motion for a fixed particle then depends upon the vertical co-ordinate z_0. Thus the mass transport can only be defined, in general, if terms of higher order than the second are neglected, that is, for small motions. We shall therefore define the mass-transport velocity as being that given by equation (21).

The mass-transport velocity may be measured as the ratio of the displacement $\mathbf{d}$ of a particle to the length t of the corresponding time interval provided that $|\mathbf{d}| \ll l$ and that the contribution to $\mathbf{d}$ from the second-order terms is large compared with that from the first-order terms; this implies $|\epsilon^2 \overline{\mathbf{U}_2} t| \gg |\epsilon \mathbf{U}_1 \tau|$ and so $t \gg \epsilon^{-1}\tau$. Both conditions are satisfied if ϵ is sufficiently small and if t is of order, say, $\epsilon^{-\frac{3}{2}}\tau$.

Let $\mathbf{f}$ be any periodic quantity associated with the motion and let

$$\mathbf{f} = \epsilon \mathbf{f}_1 + \epsilon^2 \mathbf{f}_2 + \dots, \tag{22}$$

where $\overline{\mathbf{f}_1}$ is zero. Then we may show similarly that the mean value of $\mathbf{f}$ following a particle is given by

$$\epsilon^2\left(\overline{\mathbf{f}_2} + \overline{\int \mathbf{u}_1\,\mathrm{d}t \cdot \operatorname{grad} \mathbf{f}_1}\right) \tag{23}$$

to the second order of approximation. Suppose $\mathbf{f}$ is the acceleration,

$$\mathbf{f} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\operatorname{grad}\mathbf{u}; \tag{24}$$

then

$$\mathbf{f}_1 = \frac{\partial \mathbf{u}_1}{\partial t}, \quad \mathbf{f}_2 = \frac{\partial \mathbf{u}_2}{\partial t} + \mathbf{u}_1\cdot\operatorname{grad}\mathbf{u}_1, \tag{25}$$

and the mean value of $\mathbf{f}$ following a particle is therefore given by

$$\begin{aligned} &\epsilon^2\left(\overline{\frac{\partial \mathbf{u}_2}{\partial t}} + \overline{\mathbf{u}_1\cdot\operatorname{grad}\mathbf{u}_1} + \overline{\int \mathbf{u}_1\,\mathrm{d}t\cdot\operatorname{grad}\frac{\partial \mathbf{u}_1}{\partial t}}\right), \\ &= \epsilon^2\left(\overline{\frac{\partial \mathbf{u}_2}{\partial t}} + \frac{\partial}{\partial t}\left\{\overline{\int \mathbf{u}_1\,\mathrm{d}t\cdot\operatorname{grad}\mathbf{u}_1}\right\}\right), \\ &= \frac{\epsilon^2}{\tau}\left[\mathbf{u}_2 + \int \mathbf{u}_1\,\mathrm{d}t\cdot\operatorname{grad}\mathbf{u}_1\right]_{t=0}^{t=\tau}, \end{aligned} \tag{26}$$

which vanishes by the periodicity of the motion.

The mean acceleration of a particle is therefore of a higher order than the second. This, indeed, is what we should expect. For the mean acceleration over one complete period is the difference between the initial and final velocities, divided by τ. But since in this time the particle has advanced through a distance of second order, the difference between the velocities at the initial and final positions of the particle is of third order at most.

3. The stream function for $\overline{\mathbf{U}}$

In the following we shall restrict ourselves to the consideration of two-dimensional motion only; thus if u, v and w are the components of the velocity, v is zero, and u and w are independent of the horizontal co-ordinate y. Assuming the fluid to be incompressible we have

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{27}$$

whence

$$(u, w) = \left(\frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial x}\right), \tag{28}$$

where ψ is a stream function. The vorticity is given by

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \nabla^2\psi. \tag{29}$$

We may write

$$\left.\begin{aligned} (u, w) &= \epsilon(u_1, w_1) + \epsilon^2(u_2, w_2) + \dots, \\ \psi &= \epsilon\psi_1 + \epsilon^2\psi_2 + \dots, \end{aligned}\right\} \tag{30}$$

so that

$$\left.\begin{aligned} &\frac{\partial u_i}{\partial x} + \frac{\partial w_i}{\partial z} = 0, \\ &(u_i, w_i) = \left(\frac{\partial \psi_i}{\partial z}, -\frac{\partial \psi_i}{\partial x}\right), \\ &\frac{\partial u_i}{\partial z} - \frac{\partial w_i}{\partial x} = \nabla^2\psi_i, \end{aligned}\right\} \quad (i = 1, 2, \dots). \tag{31}$$

From (12), the arbitrary function of the time contained in ψ_1 may be chosen so that

$$\overline{\psi_1} = 0. \tag{32}$$

The components $(\epsilon^2 \overline{U_2}, \epsilon^2 \overline{W_2})$ of the mass-transport velocity are given by

$$\left.\begin{aligned} \overline{U_2} &= \overline{u_2} + \overline{\int u_1 \,\mathrm{d}t \frac{\partial u_1}{\partial x}} + \overline{\int w_1 \,\mathrm{d}t \frac{\partial u_1}{\partial z}}, \\ \overline{W_2} &= \overline{w_2} + \overline{\int u_1 \,\mathrm{d}t \frac{\partial w_1}{\partial x}} + \overline{\int w_1 \,\mathrm{d}t \frac{\partial w_1}{\partial z}}. \end{aligned}\right\} \tag{33}$$

Now if A and B denote any periodic quantities we have identically

$$\overline{\frac{\partial A}{\partial t} B + A \frac{\partial B}{\partial t}} = \overline{\frac{\partial}{\partial t}(AB)} = \frac{1}{\tau}[AB]_{t=0}^{t=\tau} = 0. \tag{34}$$

Hence

$$\left.\begin{aligned} \overline{U_2} &= \overline{u_2} + \overline{\int \frac{\partial \psi_1}{\partial z} \mathrm{d}t \frac{\partial^2 \psi_1}{\partial x \partial z}} - \overline{\int \frac{\partial \psi_1}{\partial x} \mathrm{d}t \frac{\partial^2 \psi_1}{\partial z^2}} = \frac{\partial \Psi}{\partial z}, \\ \overline{W_2} &= \overline{w_2} - \overline{\int \frac{\partial \psi_1}{\partial z} \mathrm{d}t \frac{\partial^2 \psi_1}{\partial x^2}} + \overline{\int \frac{\partial \psi_1}{\partial x} \mathrm{d}t \frac{\partial^2 \psi_1}{\partial x \partial z}} = -\frac{\partial \Psi}{\partial x}, \end{aligned}\right\} \tag{35}$$

where

$$\Psi = \overline{\psi_2} + \overline{\int \frac{\partial \psi_1}{\partial z} \mathrm{d}t \frac{\partial \psi_1}{\partial x}}. \tag{36}$$

Thus $\epsilon^2 \Psi$ is a stream function for the mass-transport velocity $\overline{\mathbf{U}}$.

4. The equations of motion

The equations of motions for a viscous, incompressible fluid may be written

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} - \nu \nabla^2\right)(u, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\left(\frac{p}{\rho} - gz\right) \tag{37}$$

in the usual notation. On differentiating the first component of (37) with respect to z and the second with respect to x, and subtracting, we find

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} - \nu \nabla^2\right) \nabla^2 \psi = 0, \tag{38}$$

and hence

$$\overline{\left(u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} - \nu \nabla^2\right) \nabla^2 \psi} = 0. \tag{39}$$

The second and third terms in equation (38) represent minus the rate of change of the vorticity at a fixed point due to convection; the last term, which is similar to a term in the equation of heat conduction, represents minus the rate of change of the vorticity due to viscous diffusion. On substituting from equations (30) and formally equating the coefficient of the highest power of ϵ to zero we have, from (38),

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 \psi_1 = 0, \tag{40}$$

and from (39)

$$\overline{\left(u_1 \frac{\partial}{\partial x} + w_1 \frac{\partial}{\partial z}\right) \nabla^2 \psi_1} - \nu \nabla^4 \overline{\psi_2} = 0. \tag{41}$$

Equation (40) gives
$$\nabla^2\psi_1 = \nu\int\nabla^4\psi_1\,\mathrm{d}t, \tag{42}$$
so that on substitution in (41) we have
$$\nabla^4\overline{\psi_2} = \overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\int\nabla^4\psi_1\,\mathrm{d}t}. \tag{43}$$
Hence the field equation for Ψ in terms of ψ_1 is
$$\nabla^4\Psi = \overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\int\nabla^4\psi_1\,\mathrm{d}t}+\nabla^4\overline{\int\frac{\partial\psi_1}{\partial z}\,\mathrm{d}t\,\frac{\partial\psi_1}{\partial x}}. \tag{44}$$

The introduction of viscous terms into the equations of motion involves a new fundamental length δ, $\equiv(2\nu/\sigma)^{\frac{1}{2}}$, and a new dimensionless ratio a/δ. In the case of water waves, if $\nu = 0{\cdot}01\,\mathrm{cm^2/s}$ and $\tau = 1{\cdot}0\,\mathrm{s}$, we see that δ is of the order of $0{\cdot}02\,\mathrm{cm}$. We may therefore assume that
$$\delta/l \ll 1 \tag{45}$$
(but not necessarily that $a/\delta\ll 1$). Now, a typical periodic solution of the equations
$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)f = 0, \quad \bar{f} = 0 \tag{46}$$
is given by
$$f_0 = \mathrm{e}^{\mathrm{i}(k_1x+k_2z+n\sigma t)}, \tag{47}$$
where n is a positive integer and
$$k_1^2+k_2^2 = -\mathrm{i}n\sigma/\nu = -\mathrm{i}n/\delta^2. \tag{48}$$
Hence, in a direction perpendicular to the plane
$$\mathscr{R}(\mathrm{i}k_1x+\mathrm{i}k_2z) = 0 \tag{49}$$
f_0 must increase or decrease by a factor e in a distance of the order of δ. If f_0 is to remain bounded in the interior of the fluid, it can be appreciably large only in the neighbourhood of the boundaries, and must decrease inwards exponentially.

It is useful to distinguish between the 'boundary layer' or the region near the boundaries whose thickness is of the order of δ, and the 'interior' of the fluid, or the region 'beyond', i.e. inside, the boundary layer. For the remainder of the present section we shall be concerned only with the motion in the interior.

From equations (32) and (40) we see that $\nabla^2\psi_1$ satisfies equations (46). Therefore, assuming that $\nabla^2\psi_1$ is expressible as the sum of functions of the type (47) over any region of the interior, we may expect that
$$\nabla^2\psi_1 \to 0 \tag{50}$$
exponentially inwards. The second-order terms in equation (38) now give
$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2\psi_2 \to 0 \tag{51}$$
in the interior, so that by a similar argument we may expect that
$$\nabla^2(\psi_2-\overline{\psi_2}) \to 0 \tag{52}$$
exponentially inwards. Equation (50) states that in the interior the first-order vorticity is exponentially small, while equation (52) states that the second-order vorticity becomes

independent of the time, though it is not necessarily zero. From the third-order terms in (38) we now have

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2\psi_3+\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\nabla^2\psi_2}\to 0, \tag{53}$$

so that

$$\nabla^2\psi_3=-\left(\int u_1\,\mathrm{d}t\frac{\partial}{\partial x}+\int w_1\,\mathrm{d}t\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}+\nabla^2\psi_3', \tag{54}$$

where

$$\nabla^2(\psi_3'-\overline{\psi_3'})\to 0. \tag{55}$$

Let us now return to equation (39) and retain temporarily all the terms up to the fourth order. Assuming (50) and (52) we have, in the interior,

$$\epsilon^4\left[\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\nabla^2\psi_3+\left(\overline{u_2}\frac{\partial}{\partial x}+\overline{w_2}\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}}\right]-\nu\nabla^4(\epsilon^2\overline{\psi_2}+\epsilon^3\overline{\psi_3}+\epsilon^4\overline{\psi_4})=0. \tag{56}$$

We may substitute for $\nabla^2\psi_3$ from equation (54). Then since

$$\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\nabla^2\psi_3'}\to\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)}\nabla^2\overline{\psi_3'}=0, \tag{57}$$

in the interior of the fluid we have

$$\epsilon^4\left[-\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\left\{\left(\int u_1\,\mathrm{d}t\frac{\partial}{\partial x}+\int w_1\,\mathrm{d}t\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}\right\}}+\left(\overline{u_2}\frac{\partial}{\partial x}+\overline{w_2}\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}\right]$$
$$-\nu\nabla^4(\epsilon^2\overline{\psi_2}+\epsilon^3\overline{\psi_3}+\epsilon^4\overline{\psi_4})=0. \tag{58}$$

Now if A, B and C are any three periodic quantities we have identically

$$\overline{A\frac{\partial}{\partial x}\left(B\frac{\partial C}{\partial x}\right)}=\overline{A\frac{\partial B}{\partial x}\frac{\partial C}{\partial x}}+\overline{AB\frac{\partial^2 C}{\partial x^2}}. \tag{59}$$

But, if $A=\partial B/\partial t$, and C is independent of the time,

$$\overline{AB\frac{\partial^2 C}{\partial x^2}}=\overline{B\frac{\partial B}{\partial t}}\frac{\partial^2 C}{\partial x^2}=\frac{1}{2\tau}[B^2]_0^\tau\frac{\partial^2 C}{\partial x^2}=0. \tag{60}$$

Thus writing

$$A=u_1,\quad B=\int u_1\,\mathrm{d}t,\quad C=\nabla^2\psi_2=\nabla^2\overline{\psi_2}, \tag{61}$$

we have

$$\overline{u_1\frac{\partial}{\partial x}\left(\int u_1\,\mathrm{d}t\frac{\partial}{\partial x}\nabla^2\psi_2\right)}=\left(\overline{u_1\frac{\partial}{\partial x}\int u_1\,\mathrm{d}t}\right)\frac{\partial}{\partial x}\nabla^2\overline{\psi_2}, \tag{62}$$

and therefore (using equation (34))

$$\overline{u_1\frac{\partial}{\partial x}\left(\int u_1\,\mathrm{d}t\frac{\partial}{\partial x}\nabla^2\psi_2\right)}=-\left(\overline{\int u_1\,\mathrm{d}t\frac{\partial u_1}{\partial x}}\right)\frac{\partial}{\partial x}\nabla^2\overline{\psi_2}. \tag{63}$$

Similarly

$$\overline{w_1\frac{\partial}{\partial z}\left(\int w_1\,\mathrm{d}t\frac{\partial}{\partial z}\nabla^2\psi_2\right)}=-\left(\overline{\int w_1\,\mathrm{d}t\frac{\partial w_1}{\partial z}}\right)\frac{\partial}{\partial z}\nabla^2\overline{\psi_2} \tag{64}$$

and

$$\overline{u_1\frac{\partial}{\partial x}\left(\int w_1\,\mathrm{d}t\frac{\partial}{\partial z}\nabla^2\psi_2\right)}+\overline{w_1\frac{\partial}{\partial z}\left(\int u_1\,\mathrm{d}t\frac{\partial}{\partial x}\nabla^2\psi_2\right)}$$
$$=-\left(\overline{\int u_1\,\mathrm{d}t\frac{\partial w_1}{\partial x}}\right)\frac{\partial}{\partial z}\nabla^2\overline{\psi_2}-\left(\overline{\int w_1\,\mathrm{d}t\frac{\partial u_1}{\partial z}}\right)\frac{\partial}{\partial x}\nabla^2\overline{\psi_2}. \tag{65}$$

Therefore on substitution in equation (58) we find

$$\epsilon^4\left(\overline{U_2}\frac{\partial}{\partial x}+\overline{W_2}\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}-\nu\nabla^4(\epsilon^2\overline{\psi_2}+\epsilon^3\overline{\psi_3}+\epsilon^4\overline{\psi_4})=0, \tag{66}$$

where $\overline{U_2}$ and $\overline{W_2}$ are given by (35). The first group of terms on the left-hand side represents minus the rate of change of the vorticity due to convection by the mass-transport velocity. The second group of terms represents minus the rate of change of the vorticity due to viscous 'conduction'. These groups of terms may be called the convection terms and the conduction terms respectively.

Suppose that all terms in (66) of higher order than the second are neglected. We then have

$$\nabla^4\overline{\psi_2}=0, \tag{67}$$

or

$$\nabla^4\Psi=\nabla^4\overline{\int\frac{\partial\psi_1}{\partial z}\,\mathrm{d}t\,\frac{\partial\psi_1}{\partial x}}, \tag{68}$$

which are the equations that would be obtained by setting $\nabla^2\psi_1=0$ in the right-hand side of equations (43) and (44). But, if the velocity gradients in the interior of the fluid are not large, then the ratio of the convection terms to the conduction terms in equation (66) is of order $\epsilon^2\sigma/\nu$, that is, of order a^2/δ^2. Therefore a *necessary* condition for the validity of equations (67) and (68) in the interior of the fluid is that $a^2/\delta^2\ll 1$.

In most practical cases, however, we shall have $a\gg\delta$; so that the 'conduction equation' (67) will not apply. We should expect in this case that the appropriate field equation would be that obtained by equating to zero the convection terms on the left-hand side of (66). This cannot be deduced from the preceding analysis, which rests on the assumption that $\alpha^2/\delta^2\ll 1$; but the same equation can be derived by another method. Let us assume that the viscous terms in the original equation of motion (38) are entirely negligible; thus

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right)\nabla^2\psi=0 \tag{69}$$

and

$$\overline{\left(u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right)\nabla^2\psi}=0. \tag{70}$$

On substituting from (30) and equating coefficients of ϵ successively to zero we have from (69), in the first approximation,

$$\frac{\partial}{\partial t}(\nabla^2\psi_1)=0, \tag{71}$$

so that $\nabla^2\psi_1$ is independent of t. Thus, by (32),

$$\nabla^2\psi_1=\nabla^2\overline{\psi_1}=0. \tag{72}$$

From the second-order terms in (69)

$$\frac{\partial}{\partial t}(\nabla^2\psi_2)+\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\nabla^2\psi_1=0. \tag{73}$$

Since the second group of terms vanishes,

$$\frac{\partial}{\partial t}(\nabla^2\psi_2)=0, \tag{74}$$

and so

$$\nabla^2\psi_2=\nabla^2\overline{\psi_2}. \tag{75}$$

Similarly in the third approximation

$$\frac{\partial}{\partial t}(\nabla^2\psi_3)+\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}=0, \tag{76}$$

so that

$$\nabla^2\psi_3=-\left\{\int u_1\,\mathrm{d}t\frac{\partial}{\partial x}+\int w_1\,\mathrm{d}t\frac{\partial}{\partial z}\right\}\nabla^2\overline{\psi_2}. \tag{77}$$

The terms of highest order in (70) give

$$\overline{\left(u_1\frac{\partial}{\partial x}+w_1\frac{\partial}{\partial z}\right)\nabla^2\psi_3}+\left(\overline{u_2}\frac{\partial}{\partial x}+\overline{w_2}\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}=0. \tag{78}$$

On substituting from (78) into (77) and using the relations (63), (64) and (65) we obtain

$$\left(\overline{U_2}\frac{\partial}{\partial x}+\overline{W_2}\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}=0, \tag{79}$$

or, from (35) and (36),

$$\left(\frac{\partial\Psi}{\partial z}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\left(\Psi-\overline{\int\frac{\partial\psi_1}{\partial z}\,\mathrm{d}t\frac{\partial\psi_1}{\partial x}}\right)=0. \tag{80}$$

In deriving (79) the first non-zero term omitted owing to the neglect of the viscosity is $\epsilon^2\nu\nabla^4\psi_2$ (since $\nabla^4\psi_1$ is zero); the largest terms retained are the fourth-order terms in equation (79). Hence a *necessary* condition for the validity of these equations is that

$$\epsilon^2\nu\nabla^4\psi_2\ll\epsilon^4\left(\overline{U_2}\frac{\partial}{\partial x}+\overline{W_2}\frac{\partial}{\partial z}\right)\nabla^2\overline{\psi_2}, \tag{81}$$

and hence that $a^2\gg\delta^2$.

Equations (68) and (80) may be called the 'conduction equation' and the 'convection equation' respectively.

Let us consider equation (79) more closely. It may be written in the form

$$\frac{\partial(\Psi,\nabla^2\overline{\psi_2})}{\partial(x,z)}=0, \tag{82}$$

where the left-hand side is the Jacobian of Ψ and $\nabla^2\overline{\psi_2}$. Thus $\nabla^2\overline{\psi_2}$ is functionally related to Ψ:

$$\nabla^2\overline{\psi_2}=F(\Psi), \tag{83}$$

or by (36)

$$\nabla^2\left(\Psi-\overline{\int\frac{\partial\psi_1}{\partial z}\,\mathrm{d}t\frac{\partial\psi_1}{\partial x}}\right)=F(\Psi). \tag{84}$$

Hence the vorticity is constant along a stream-line. Also, since ψ_1 satisfies Laplace's equation we have, on differentiation,

$$\begin{aligned}\nabla^2\overline{\int\frac{\partial\psi_1}{\partial z}\,\mathrm{d}t\frac{\partial\psi_1}{\partial x}}&=2\overline{\int\frac{\partial^2\psi_1}{\partial x\partial z}\,\mathrm{d}t\frac{\partial^2\psi_1}{\partial x^2}}+2\overline{\int\frac{\partial^2\psi_1}{\partial z^2}\,\mathrm{d}t\frac{\partial^2\psi_1}{\partial x\partial z}}\\&=2\overline{\int\frac{\partial^2\psi_1}{\partial x\partial z}\,\mathrm{d}t\frac{\partial^2\psi_1}{\partial x^2}}-2\overline{\int\frac{\partial^2\psi_1}{\partial x^2}\,\mathrm{d}t\frac{\partial^2\psi_1}{\partial x\partial z}}\\&=-4\overline{\int\frac{\partial^2\psi_1}{\partial x^2}\,\mathrm{d}t\frac{\partial^2\psi_1}{\partial x\partial z}},\end{aligned} \tag{85}$$

using equation (34). From (84) and (85) we obtain

$$\nabla^2\Psi=F(\Psi)-4\overline{\int\frac{\partial^2\psi_1}{\partial x^2}\,\mathrm{d}t\frac{\partial^2\psi_1}{\partial x\partial z}}, \tag{86}$$

an alternative form of the conduction equation.

PART II. THE BOUNDARY LAYERS

5. Introduction

In part I the mass-transport velocity in any oscillatory motion was defined, and field equations were obtained for the mass-transport stream function Ψ in the interior. It was shown, however, that the neighbourhood of the boundaries requires special consideration, on account of the large velocity gradients encountered there; it can no longer be assumed, for example, that the first-order vorticity is zero, as in the interior of the fluid.

An exact solution of the problem of an oscillating plane boundary, in a fluid at rest at infinity, was given by Stokes (1851). Lamb (1932, p. 662) gave the solution to the closely related problem of a semi-infinite fluid, with a fixed plane boundary, moving under the action of a harmonically oscillating body force. In these exact solutions the vorticity remains always in the neighbourhood of the boundaries, and the motion beyond a layer of thickness of the order of δ, $= (2\nu/\sigma)^{\frac{1}{2}}$, is zero. Also the mass-transport velocity vanishes identically. Approximate solutions for wave motion in water of finite or infinite depth have been given by Basset (1888), Hough (1896) and Lamb (1932). In these approximate solutions the vorticity is also confined to the boundaries, to the first approximation; but to obtain the mass transport it is necessary to study the second-order terms.

The objections to a direct extension of the solutions of Basset and Hough to a second approximation have been discussed in part I. Briefly, the method would only be valid for very small values of the ratio a/δ, where a is the wave amplitude. A different method, for the case of a circular cylinder oscillating in an infinite fluid, was used by Schlichting (1932). This involved initial neglect of δ/l, where l was the radius of the cylinder—essentially a boundary-layer approximation. It will be found that in Schlichting's analysis there is no implied restriction on the ratio a/δ for the motion near the boundaries.* In the following we shall use a similar approximation to Schlichting's, but treat a much more general problem, assuming an arbitrary oscillating motion of the boundaries, and taking into consideration more than one type of boundary condition.

6. Co-ordinates and general equations

Since the normal displacement of the boundary may be large compared with δ, a co-ordinate system must be chosen which is attached to the moving boundary. As in part I, assume the motion to be two-dimensional and independent of y and let

s = arc length measured along the boundary,

n = distance measured inwards along a normal,

$\kappa(s,t)$ = curvature of the boundary (positive when concave inwards),

(see figure 2a). The co-ordinates (s, n) are to be chosen so as to be in the same sense, right-handed or left-handed, as the cartesian co-ordinates (x, z) of part I. (s, n) are orthogonal,

* However, for the interior of the fluid Schlichting uses the 'conduction equation' (see §4), which may not be justifiable.

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the lines n = constant, being parallel curves. The square of the displacement corresponding to small increments $\mathrm{d}s$, $\mathrm{d}n$ is

$$\eta^2 \mathrm{d}s^2 + \mathrm{d}n^2, \tag{87}$$

where

$$\eta = 1 - n\kappa. \tag{88}$$

If q_s and q_n denote the components of velocity, resolved parallel to the directions of s and n increasing, the equation of continuity

$$\frac{\partial q_s}{\partial s} + \frac{\partial}{\partial n}(\eta q_n) = 0 \tag{89}$$

implies the existence of a stream function ψ such that

$$(q_s, q_n) = \left(\frac{\partial \psi}{\partial n}, -\frac{1}{\eta}\frac{\partial \psi}{\partial s}\right). \tag{90}$$

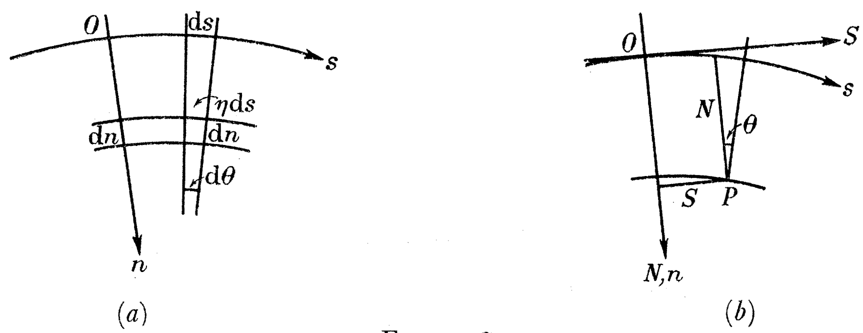

FIGURE 2

To find the normal and tangential stresses we temporarily introduce rectangular coordinates (S, N) tangential to (s, n) at the origin. The corresponding velocity components q_S and q_N are given by

$$\left.\begin{aligned} q_S &= q_s \cos\theta - q_n \sin\theta, \\ q_N &= q_s \sin\theta + q_n \cos\theta, \end{aligned}\right\} \tag{91}$$

where θ is the angle between the normals $s = 0$ and s = constant (see figure 2b). Thus

$$\theta = \int_0^s \kappa \,\mathrm{d}s, \qquad \frac{\partial \theta}{\partial s} = \kappa, \qquad \frac{\partial \theta}{\partial n} = 0. \tag{92}$$

When $s = 0 = \theta$ we have

$$\left.\begin{aligned} q_S &= q_s, & \frac{\partial q_S}{\partial s} &= \frac{\partial q_s}{\partial s} - \kappa q_n, & \frac{\partial q_S}{\partial n} &= \frac{\partial q_s}{\partial n}, \\ q_N &= q_n, & \frac{\partial q_N}{\partial s} &= \frac{\partial q_n}{\partial s} + \kappa q_s, & \frac{\partial q_N}{\partial n} &= \frac{\partial q_n}{\partial n}. \end{aligned}\right\} \tag{93}$$

The normal stress p_{nn} and the tangential stress p_{ns} are given, when $s = 0$, by

$$\left.\begin{aligned} \frac{1}{\rho\nu}(p_{nn} + p) &= 2\frac{\partial q_N}{\partial N} = 2\frac{\partial q_N}{\partial n}, \\ \frac{1}{\rho\nu} p_{ns} &= \frac{\partial q_S}{\partial N} + \frac{\partial q_N}{\partial S} = \frac{\partial q_S}{\partial n} + \frac{1}{\eta}\frac{\partial q_N}{\partial s}, \end{aligned}\right\} \tag{94}$$

where p is the mean pressure. Thus from (93)

$$\left.\begin{aligned} \frac{1}{\rho\nu}(p_{nn}+p) &= -2\frac{\partial}{\partial n}\left(\frac{1}{\eta}\frac{\partial\psi}{\partial s}\right), \\ \frac{1}{\rho\nu}p_{ns} &= \frac{\partial^2\psi}{\partial n^2}+\frac{1}{\eta}\left[\kappa\frac{\partial\psi}{\partial n}-\frac{\partial}{\partial s}\left(\frac{1}{\eta}\frac{\partial\psi}{\partial s}\right)\right]. \end{aligned}\right\} \tag{95}$$

Since the form of these equations is independent of the position of the origin, they are valid for all values of (s, n).

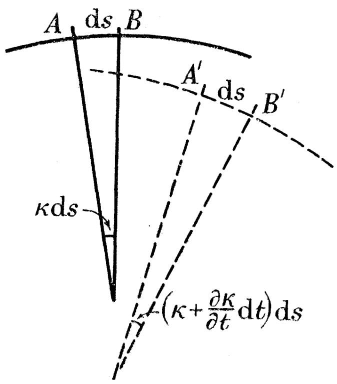

FIGURE 3

To describe the motion of the co-ordinate system, let

$V_s(s,t)$ = velocity of the point $(s, 0)$ parallel to the boundary,
$V_n(s,t)$ = velocity of the point $(s, 0)$ normal to the boundary,
$\Omega(s,t)$ = angular velocity of the normal s = constant (positive in the sense of θ increasing).

Then the velocity components of the point having co-ordinates (s, n) are

$$(V_s - n\Omega, V_n), \tag{96}$$

and if $(\dot{s}, \dot{n})$ denotes the rate at which the co-ordinates of a particular element of fluid are increasing, and (q_s, q_n) denotes its actual velocity in space, we have

$$\dot{s} = \frac{1}{\eta}(q_s - V_s + n\Omega), \quad \dot{n} = q_n - V_n. \tag{97}$$

The following relations between V_n, V_s and Ω will be of use:

$$\left.\begin{aligned} \frac{\partial V_s}{\partial s} - \kappa V_n &= 0, \\ \frac{\partial V_n}{\partial s} + \kappa V_s &= \Omega, \end{aligned}\right\} \tag{98}$$

$$\frac{\partial \kappa}{\partial t} = \frac{\partial \Omega}{\partial s}. \tag{99}$$

These may be proved as follows. Consider the normals to the surface at two neighbouring points A and B, separated by an arc-length $\mathrm{d}s$ (see figure 3). Suppose that, in a short time $\mathrm{d}t$, A and B are displaced to A' and B' respectively. The displacements of A perpendicular and parallel to the normal at A are $V_s\,\mathrm{d}t$ and $V_n\,\mathrm{d}t$; thus, if the tangent and normal at A are taken as co-ordinate axes, the vector $\overrightarrow{AA'}$ is given by

$$\overrightarrow{AA'} = (V_s, V_n)\,\mathrm{d}t. \tag{100}$$

Similarly, the displacements of B perpendicular and parallel to the normal at B are

$$\left(V_s+\frac{\partial V_s}{\partial s}\mathrm{d}s\right)\mathrm{d}t \quad \text{and} \quad \left(V_n+\frac{\partial V_n}{\partial s}\mathrm{d}s\right)\mathrm{d}t.$$

But the normal at B makes an angle $\kappa\,\mathrm{d}s$ with the normal at A. Hence, referred to the tangent and normal at A, we have

$$\overrightarrow{BB'} = \left(V_s+\frac{\partial V_s}{\partial s}\mathrm{d}s-\kappa\,\mathrm{d}s V_n,\ V_n+\frac{\partial V_n}{\partial s}\mathrm{d}s+\kappa\,\mathrm{d}s V_s\right)\mathrm{d}t \tag{101}$$

to the present order of approximation. From (100) and (101)

$$\overrightarrow{BB'}-\overrightarrow{AA'} = \left(\frac{\partial V_s}{\partial s}-\kappa V_n,\ \frac{\partial V_n}{\partial s}+\kappa V_s\right)\mathrm{d}s\,\mathrm{d}t. \tag{102}$$

Now in time $\mathrm{d}t$ the vector $\overrightarrow{AB}$ remains of constant length $\mathrm{d}s$ (neglecting $\mathrm{d}s^3$) and turns through a small angle $\Omega\,\mathrm{d}t$. The displacement of the vector AB is therefore given by

$$\overrightarrow{A'B'}-\overrightarrow{AB} = (0, \Omega\,\mathrm{d}t\,\mathrm{d}s). \tag{103}$$

But since

$$\overrightarrow{AA'}+\overrightarrow{A'B'} = \overrightarrow{AB'} = \overrightarrow{AB}+\overrightarrow{BB'}, \tag{104}$$

the left-hand sides of (102) and (103) are equal. This proves equations (98). Equation (99) may be proved similarly: if $\widehat{AB}$, $\widehat{A'B'}$, etc., denote the angles between the normals at A and B, A' and B', etc., we have

$$\widehat{AB} = \kappa\,\mathrm{d}s, \qquad \widehat{A'B'} = \left(\kappa+\frac{\partial\kappa}{\partial t}\mathrm{d}t\right)\mathrm{d}s. \tag{105}$$

Also

$$\widehat{AA'} = \Omega\,\mathrm{d}t, \qquad \widehat{BB'} = \left(\Omega+\frac{\partial\Omega}{\partial s}\mathrm{d}s\right)\mathrm{d}t. \tag{106}$$

But

$$\widehat{AA'}+\widehat{A'B'} = \widehat{AB'} = \widehat{AB}+\widehat{BB'}, \tag{107}$$

from which (99) follows.

As a result of the first of equations (98) we may write

$$V_s = \frac{\partial\psi^{(b)}}{\partial n}, \qquad V_n = -\frac{1}{\eta}\frac{\partial\psi^{(b)}}{\partial s}, \tag{108}$$

where

$$\psi^{(b)} = -\int \eta V_n\,\mathrm{d}s, \tag{109}$$

and therefore

$$q_s-V_s = \frac{\partial\psi'}{\partial n}, \qquad q_n-V_n = -\frac{1}{\eta}\frac{\partial\psi'}{\partial s}, \tag{110}$$

where

$$\psi' = \psi-\psi^{(b)} = \psi+\int\eta V_n\,\mathrm{d}s. \tag{111}$$

Clearly $\psi^{(b)}$ is a stream function for the motion of the boundary itself; ψ' is a stream function for the motion 'relative to the boundary'. There is in general no stream function for the motion of the co-ordinate system at points other than on the boundary, since the term $(-n\Omega, 0)$ in (96) represents a divergent velocity field. From (97) and (110) we have

$$\dot{s} = \frac{1}{\eta}\left(\frac{\partial \psi'}{\partial n} + n\Omega\right), \quad \dot{n} = -\frac{1}{\eta}\frac{\partial \psi'}{\partial s}. \tag{112}$$

Consider now the equations of motion. Equation (38), which is obtained after elimination of the mean pressure p, may be expressed in the invariant form

$$\left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu\nabla^2\right)\omega = 0, \tag{113}$$

where $\mathrm{D}/\mathrm{D}t$ denotes differentiation following the motion, ∇^2 is Laplace's operator and ω is the vorticity of the fluid:

$$\omega = \nabla^2\psi. \tag{114}$$

With the present co-ordinates

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \dot{s}\frac{\partial}{\partial s} + \dot{n}\frac{\partial}{\partial n} \tag{115}$$

and

$$\nabla^2 \equiv \frac{1}{\eta}\left[\frac{\partial}{\partial s}\left(\frac{1}{\eta}\frac{\partial}{\partial s}\right) + \frac{\partial}{\partial n}\eta\left(\frac{\partial}{\partial n}\right)\right]. \tag{116}$$

Thus (113) can be written

$$\left[\frac{\partial}{\partial t} + \frac{1}{\eta}\left(\frac{\partial \psi'}{\partial n} + n\Omega\right) - \frac{1}{\eta}\frac{\partial \psi'}{\partial s}\frac{\partial}{\partial n} - \nu\nabla^2\right]\nabla^2\psi = 0, \tag{117}$$

∇^2 being given by (116).

For future reference it will be convenient to state also the equation of motion for each component of the velocity separately. If the co-ordinates (S, N) are taken to refer to a definite instant of time, say $t = 0$, then we have

$$\left.\begin{aligned} \left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu\nabla^2\right) q_S &= -\frac{\partial}{\partial S}\left(\frac{p}{\rho} - gz\right), \\ \left(\frac{\mathrm{D}}{\mathrm{D}t} - \nu\nabla^2\right) q_N &= -\frac{\partial}{\partial N}\left(\frac{p}{\rho} - gz\right). \end{aligned}\right\} \tag{118}$$

Now when $t = 0$,

$$\left.\begin{aligned} \frac{\partial q_S}{\partial t} &= \frac{\partial q_s}{\partial t} - \Omega q_n, \\ \frac{\partial q_N}{\partial t} &= \frac{\partial q_n}{\partial t} + \Omega q_s. \end{aligned}\right\} \tag{119}$$

Also, from equations (91) we find

$$\left.\begin{aligned} \frac{\partial^2 q_S}{\partial s^2} &= \frac{\partial^2 q_s}{\partial s^2} - \kappa^2 q_s - 2\kappa\frac{\partial q_n}{\partial s} - \frac{\mathrm{d}\kappa}{\mathrm{d}s}q_n, \quad & \frac{\partial^2 q_S}{\partial n^2} &= \frac{\partial^2 q_s}{\partial n^2}, \\ \frac{\partial^2 q_N}{\partial s^2} &= \frac{\partial^2 q_n}{\partial s^2} - \kappa^2 q_n + 2\kappa\frac{\partial q_s}{\partial s} + \frac{\mathrm{d}\kappa}{\mathrm{d}s}q_s, \quad & \frac{\partial^2 q_N}{\partial n^2} &= \frac{\partial^2 q_n}{\partial n^2}, \end{aligned}\right\} \tag{120}$$

whence, after some simplification, we have

$$\left.\begin{aligned} \nabla^2 q_S &= \frac{\partial^2 q_s}{\partial n^2} - \frac{1}{\eta}\frac{\partial^2 q_n}{\partial s\,\partial n} - \frac{1}{\rho\nu}\kappa p_{ns}, \\ \nabla^2 q_N &= \frac{\partial^2 q_n}{\partial n^2} + \frac{1}{\eta}\frac{\partial^2 q_s}{\partial s\,\partial n} - \frac{1}{\rho\nu}\frac{1}{\eta}\frac{\partial p_{ns}}{\partial s}. \end{aligned}\right\} \tag{121}$$

The equations of motion are therefore

$$\left(\frac{\partial q_s}{\partial t}-\Omega q_n\right)+\dot{s}\left(\frac{\partial q_s}{\partial s}-\kappa q_n\right)+\dot{n}\frac{\partial q_s}{\partial n}-\nu\left(\frac{\partial^2 q_s}{\partial n^2}-\frac{1}{\eta}\frac{\partial^2 q_n}{\partial s\,\partial n}\right)+\frac{1}{\rho}\kappa p_{ns}=-\frac{1}{\eta}\frac{\partial}{\partial s}\left(\frac{p}{\rho}-gz\right) \tag{122}$$

and

$$\left(\frac{\partial q_n}{\partial t}+\Omega q_s\right)+\dot{s}\left(\frac{\partial q_n}{\partial s}+\kappa q_s\right)+\dot{n}\frac{\partial q_n}{\partial n}-\nu\left(\frac{\partial^2 q_n}{\partial n^2}+\frac{1}{\eta}\frac{\partial^2 q_s}{\partial s\,\partial n}\right)+\frac{1}{\rho\eta}\frac{\partial p_{ns}}{\partial s}=-\frac{\partial}{\partial n}\left(\frac{p}{\rho}-gz\right). \tag{123}$$

Up to the present point no approximations of any kind have been made.

In order to define the mass-transport velocity we assume, first, that the motion at each point of space is time-periodic, and that the co-ordinate system is also chosen in a periodic way; all functions of the velocities, for given co-ordinates (s, n), are then periodic in t. Secondly, we assume that the motion is expressible in the form of an asymptotic series

$$\psi = \epsilon\psi_1+\epsilon^2\psi_2+\dots, \tag{124}$$

and we write

$$q_s = \epsilon q_{s\,1}+\epsilon^2 q_{s\,2}+\dots, \tag{125}$$

with similar expressions for q_n, V_s, V_n, Ω, $\dot{s}$, $\dot{n}$, $\psi^{(b)}$ and ψ'. Thus the motion of the co-ordinate system, defined by V_s, V_n and Ω, is of first order at most. On the other hand, we write

$$\left.\begin{aligned}\kappa &= \kappa_0+\epsilon\kappa_1+\epsilon^2\kappa_2+\dots,\\ \eta &= \eta_0+\epsilon\eta_1+\epsilon^2\eta_2+\dots,\end{aligned}\right\} \tag{126}$$

allowing for a curvature κ_0 of the boundary in the undisturbed state. As in § 2 the mean values of the first-order velocities at each point in space are supposed to be zero. Thus, if the co-ordinates (S, N) are chosen as above, we have with an obvious notation

$$\overline{q_{S\,1}} = \overline{q_{N\,1}} = 0, \tag{127}$$

where a bar denotes mean values with respect to time. It follows from (91) that

$$\overline{q_{s\,1}} = \overline{q_{n\,1}} = 0, \tag{128}$$

since θ is constant with respect to time except possibly for a first-order variation. It may be shown also that

$$\overline{V_{s\,1}} = \overline{V_{n\,1}} = 0, \tag{129}$$

and, since

$$\overline{\Omega} = \overline{\frac{\partial\theta}{\partial t}} = [\theta]_{t=0}^{t=\tau} = 0, \tag{130}$$

we have

$$\overline{\Omega_1} = \overline{\Omega_2} = \dots = 0. \tag{131}$$

From (98) and (99) we have also the useful relations

$$\left.\begin{aligned}&\frac{\partial V_{s\,1}}{\partial s}-\kappa_0 V_{n\,1} = 0,\\ &\frac{\partial V_{n\,1}}{\partial s}+\kappa_0 V_{s\,1} = \Omega_1,\\ &\frac{\partial \kappa_1}{\partial t} = \frac{\partial \Omega_1}{\partial s}.\end{aligned}\right\} \tag{132}$$

Since, from (88),

$$\eta_0 = 1-\kappa_0 n, \qquad \eta_1 = -\kappa_1 n, \tag{133}$$

it follows, by the third of equations (132), that

$$\eta_1 = -\int \frac{\partial}{\partial s}(n\Omega_1)\,\mathrm{d}t. \tag{134}$$

The choice of the normal $s = 0$ is still at our disposal. If the boundary is rigid, the origin $(0, 0)$ may be chosen to be a point on the boundary and fixed relative to it, so that, at all points on the boundary,

$$V_s = q_s, \quad V_n = q_n. \tag{135}$$

On the other hand, it may be more convenient to take the origin at the point of intersection of the boundary with a line fixed in space, say a line normal to the position of the undisturbed surface. Since the angle between this line and the normal to the moving surface is of first order in ϵ, it follows that $V_s(0, t)$ is of order ϵ^2 at most, i.e.

$$V_{s1} = 0 \tag{136}$$

when $s = 0$. But from (132) we have

$$\frac{\partial V_{s1}}{\partial s} = \kappa_0 V_{n1}, \tag{137}$$

so that when κ_0 vanishes V_{s1} is constant along the boundary and (136) holds for all values of s. In other words, if the undisturbed surface has no curvature, the co-ordinates may be chosen so that V_{s1} vanishes at all points of the surface.

In precisely the same way as in § 2, it may be shown that the mean rate of increase of the co-ordinates (s, n) of a particle is given, to order ϵ^2, by

$$\epsilon^2\left[(\overline{\dot{s}_2}, \overline{\dot{n}_2}) + \overline{\left(\int \dot{s}_1 \,\mathrm{d}t \frac{\partial}{\partial s} + \int \dot{n}_1 \,\mathrm{d}t \frac{\partial}{\partial n}\right)(\dot{s}_1, \dot{n}_1)}\right], \tag{138}$$

where, from (112),

$$\left.\begin{aligned} \dot{s}_1 &= \frac{1}{\eta_0}\left(\frac{\partial \psi_1'}{\partial n} + n\Omega_1\right), & \dot{n}_1 &= -\frac{1}{\eta_0}\frac{\partial \psi_1'}{\partial s}, \\ \overline{\dot{s}_2} &= \frac{1}{\eta_0}\left(\overline{\frac{\partial \psi_2'}{\partial n}} - \overline{\frac{\eta_1}{\eta_0}\frac{\partial \psi_1'}{\partial n}} - \overline{\frac{\eta_1}{\eta_0} n\Omega_1}\right), & \overline{\dot{n}_2} &= -\frac{1}{\eta_0}\left(\overline{\frac{\partial \psi_2'}{\partial s}} - \overline{\frac{\eta_1}{\eta_0}\frac{\partial \psi_1'}{\partial s}}\right). \end{aligned}\right\} \tag{139}$$

But the position of the co-ordinate axes remains on the average unchanged. We therefore define the components of the mass-transport velocity $\epsilon^2(\overline{Q_{s2}}, \overline{Q_{n2}})$ by the equations

$$\left.\begin{aligned} \frac{1}{\eta_0}\overline{Q_{s2}} &= \overline{\dot{s}_2} + \overline{\int \dot{s}_1 \,\mathrm{d}t \frac{\partial \dot{s}_1}{\partial s}} + \overline{\int \dot{n}_1 \,\mathrm{d}t \frac{\partial \dot{s}_1}{\partial n}}, \\ \overline{Q_{n2}} &= \overline{\dot{n}_2} + \overline{\int \dot{s}_1 \,\mathrm{d}t \frac{\partial \dot{n}_1}{\partial s}} + \overline{\int \dot{n}_1 \,\mathrm{d}t \frac{\partial \dot{n}_1}{\partial n}}. \end{aligned}\right\} \tag{140}$$

It can be shown by direct differentiation, using equation (134) and the periodic property of the motion (equation (34)), that

$$\overline{Q_{s2}} = \frac{\partial \Psi}{\partial n}, \quad Q_{n2} = -\frac{1}{\eta_0}\frac{\partial \Psi}{\partial s}, \tag{141}$$

where Ψ, which is a stream function for the mass-transport velocity, is given by

$$\Psi = \overline{\psi_2'} + \frac{1}{\eta_0}\overline{\int\left(\frac{\partial \psi_1'}{\partial n} + n\Omega_1\right)\mathrm{d}t \frac{\partial \psi_1'}{\partial s}}, \tag{142}$$

$$= \overline{\psi_2'} - \eta_0 \overline{\int \dot{s}_1 \,\mathrm{d}t \,\dot{n}_1}. \tag{143}$$

In order to simplify the above equations we shall now make a boundary-layer approximation. The procedure we shall adopt will be, first, to neglect all quantities of order δ/l (where $\delta = (2\nu/\sigma)^{\frac{1}{2}}$ and l is a typical length associated with the geometry of the system), then to find a first-order solution in powers of ϵ, and finally to derive the mass transport. The initial neglect of δ/l involves relative errors of the order of δ/l in the first approximation. But it will be found that, although the normal velocity gradients may be of order $a\sigma/\delta$, the corresponding components of the normal velocity $\dot{n}$ relative to the boundaries are in that case of order $a\sigma\delta/l$; by equation (140), the mass transport remains a homogeneous second-order function of the velocities. Hence the relative errors involved in the mass-transport velocity are only of order δ/l; no restrictions on the ratio a/δ are implied.

The orders of magnitude of the different terms will depend, however, on the type of condition to be satisfied at the boundary. The two cases where the tangential velocity and the tangential stress, respectively, are prescribed will therefore be considered separately (§§ 7 and 8).

The velocity gradients in the interior of the fluid are assumed to be of ordinary magnitude, i.e. of order $a\sigma/l$. Thus, over distances which are small compared with l the velocity may be assumed to be uniform. The velocities or velocity gradients in the boundary layer, which may vary rapidly in the boundary layer itself, will tend to their relatively constant values 'just beyond the boundary layer', that is, in a region whose distance from the boundary is greater than a few multiples of δ but is still small compared with l. For points in this region we shall write $n = \infty$, with the understanding that this implies only $\delta \ll n \ll l$. Thus the components of the velocity just beyond the boundary layer will be denoted by $q_s^{(\infty)}$ and $q_n^{(\infty)}$; those at the surface itself by $q_s^{(0)}$ and $q_n^{(0)}$.

7. The velocities are prescribed at the boundary

When $n = 0$ we have

$$\frac{\partial \psi}{\partial s} = -V_n = -q_n^{(0)}, \qquad \frac{\partial \psi'}{\partial s} = 0, \tag{144}$$

and

$$\frac{\partial \psi}{\partial n} = q_s^{(0)}, \qquad \frac{\partial \psi'}{\partial n} = q_s^{(0)} - V_s. \tag{145}$$

Assuming the tangential velocity $\partial\psi/\partial n$ to be of order unity (in powers of δ/l) throughout the boundary layer, we have

$$\frac{\partial^2 \psi}{\partial s \partial n} = O(1), \tag{146}$$

and therefore, on integrating from $n = 0$,

$$\frac{\partial \psi}{\partial s} = -V_n + O(\delta/l), \qquad \frac{\partial \psi'}{\partial s} = O(\delta/l). \tag{147}$$

But, since the tangential velocity $q_s^{(\infty)}$ just beyond the boundary layer in general differs from $q_s^{(0)}$, we must have

$$\frac{\partial^2 \psi}{\partial n^2} = O(\delta/l)^{-1}, \qquad \frac{\partial^2 \psi}{\partial n^3} = O(\delta/l)^{-2}, \tag{148}$$

and so on. $\psi^{(b)}$ and its derivatives being of order unity at most, we have also (since $\psi' = \psi - \psi^{(b)}$)

$$\frac{\partial \psi'}{\partial n} = O(1), \qquad \frac{\partial^2 \psi'}{\partial n^2} = O(\delta/l)^{-1}, \qquad \frac{\partial^3 \psi'}{\partial n^3} = O(\delta/l)^{-2}, \tag{149}$$

etc. Each differentiation of ψ' with respect to n raises the order of magnitude by a factor $(\delta/l)^{-1}$, whereas differentiation with respect to s leaves the order of magnitude unchanged. Retaining only the terms of highest order, we have from (88) and (112)

$$\eta = 1 \tag{150}$$

and

$$\dot{s} = \frac{\partial \psi'}{\partial n}, \quad \dot{n} = -\frac{\partial \psi'}{\partial s}. \tag{151}$$

The equation of motion (117) becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial \psi'}{\partial n}\frac{\partial}{\partial s} - \frac{\partial \psi'}{\partial s}\frac{\partial}{\partial n} - \nu \frac{\partial^2}{\partial n^2}\right)\frac{\partial^2 \psi'}{\partial n^2} = 0, \tag{152}$$

and on taking mean values with respect to time we have

$$\overline{\left(\frac{\partial \psi'}{\partial n}\frac{\partial}{\partial s} - \frac{\partial \psi'}{\partial s}\frac{\partial}{\partial n} - \nu \frac{\partial^2}{\partial n^2}\right)\frac{\partial^2 \psi'}{\partial n^2}} = 0. \tag{153}$$

In the first approximation (in powers of ϵ) we have from (152)

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial n^2}\right)\frac{\partial^2 \psi'_1}{\partial n^2} = 0, \tag{154}$$

and so

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial n^2}\right)\frac{\partial \psi'_1}{\partial n} = \frac{\partial}{\partial t}(q_{s1}^{(\infty)} - V_{s\,1}), \tag{155}$$

for the expression on the left-hand side, being independent of n, equals its value just beyond the boundary layer. From (154)

$$\frac{\partial^2 \psi'_1}{\partial n^2} = \nu \int \frac{\partial^4 \psi'_1}{\partial n^4}\,\mathrm{d}t, \tag{156}$$

and from (155)

$$\frac{\partial \psi'_1}{\partial n} = \nu \int \frac{\partial^3 \psi'_1}{\partial n^3}\,\mathrm{d}t + (q_{s1}^{(\infty)} - V_{s\,1}). \tag{157}$$

The terms of lowest order in ϵ in equation (153) give

$$\overline{\left(\frac{\partial \psi'_1}{\partial n}\frac{\partial}{\partial s} - \frac{\partial \psi'_1}{\partial s}\frac{\partial}{\partial n}\right)\frac{\partial^2 \psi'_1}{\partial n^2}} - \nu \frac{\partial^4 \overline{\psi'_2}}{\partial n^4} = 0, \tag{158}$$

and so from (156)

$$\overline{\frac{\partial \psi'_1}{\partial n}\int \frac{\partial^5 \psi'_1}{\partial s \partial n^4}\,\mathrm{d}t - \frac{\partial \psi'_1}{\partial s}\int \frac{\partial^5 \psi'_1}{\partial n^5}\,\mathrm{d}t} - \frac{\partial^4 \overline{\psi'_2}}{\partial n^4} = 0. \tag{159}$$

Now from (142), to the present approximation,

$$\Psi = \overline{\psi'_2} + \overline{\int \frac{\partial \psi'_1}{\partial n}\,\mathrm{d}t\,\frac{\partial \psi'_1}{\partial s}}. \tag{160}$$

If (142) is differentiated four times with respect to n it will be seen that the terms of highest order in δ/l are simply those that would be obtained by differentiating (160) four times. Thus, by Leibniz's theorem

$$\frac{\partial^4\Psi}{\partial n^4} = \frac{\partial^4\overline{\psi_2'}}{\partial n^4}$$
$$+\overline{\int\frac{\partial^5\psi_1'}{\partial n^5}\,\mathrm{d}t\frac{\partial\psi_1'}{\partial s}}+4\overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n}}+6\overline{\int\frac{\partial^3\psi_1'}{\partial n^3}\,\mathrm{d}t\frac{\partial^3\psi_1'}{\partial s\,\partial n^2}}+4\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\frac{\partial^4\psi_1'}{\partial s\,\partial n^3}}+\overline{\int\frac{\partial\psi_1'}{\partial n}\,\mathrm{d}t\frac{\partial^5\psi_1'}{\partial s\,\partial n^4}}. \quad (161)$$

On substituting for $\partial^4\overline{\psi_2'}/\partial n^4$ from (159) and using a property of the periodicity (equation (34)), we find

$$\frac{\partial^4\Psi}{\partial n^4} = 4\overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n}}+6\overline{\int\frac{\partial^3\psi_1'}{\partial n^3}\,\mathrm{d}t\frac{\partial^3\psi_1'}{\partial s\,\partial n^2}}+4\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n^3}}. \quad (162)$$

This is our differential equation for Ψ in terms of ψ_1'. It may be integrated as follows: from (156) and (157) (and (34))

$$\overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n}}+\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\frac{\partial^4\psi_1'}{\partial s\,\partial n^3}} = \overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial}{\partial s}\left[\nu\int\frac{\partial^3\psi_1'}{\partial n^3}\,\mathrm{d}t+(q_{s1}^{(\infty)}-V_{s\,1})\right]}-\nu\overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\int\frac{\partial^3\psi_1'}{\partial s\,\partial n^3}\,\mathrm{d}t}$$
$$= \overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\infty)}-V_{s\,1})}. \quad (163)$$

Thus (162) may be written

$$\frac{\partial^4\Psi}{\partial n^4} = \overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\infty)}-V_{s\,1})}+3\frac{\partial^2}{\partial n^2}\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n}}. \quad (164)$$

On integrating twice from $n=\infty$, where $\dfrac{\partial^3\psi_1'}{\partial n^3}$, $\dfrac{\partial^2\psi_1'}{\partial n^2}$, $\dfrac{\partial^3\Psi}{\partial n^3}$ and $\dfrac{\partial^2\Psi}{\partial n^2}$ vanish to the present order, we have

$$\frac{\partial^2\Psi}{\partial n^2} = \overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\infty)}-V_{s\,1})}+3\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n}}, \quad (165)$$

which can also be written

$$\frac{\partial^2\Psi}{\partial n^2} = 4\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\infty)}-V_{s\,1})}+3\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\left(\frac{\partial^2\psi_1'}{\partial s\,\partial n}\right)_\infty^n}. \quad (166)$$

On integrating once more, from $n=0$, we have

$$\left(\frac{\partial\Psi}{\partial n}\right)_0^n = 4\overline{\int\left(\frac{\partial\psi_1'}{\partial n}\right)_0^n\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\infty)}-V_{s\,1})}+3\int_0^n\left[\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\left(\frac{\partial^2\psi_1'}{\partial s\,\partial n}\right)_\infty^n}\right]\mathrm{d}n. \quad (167)$$

Now when $n=0$ we have from (144), (145) and (160)

$$\left(\frac{\partial\Psi}{\partial n}\right)_{n=0} = \overline{(q_{s2}^{(0)}-V_{s\,2})}+\overline{\int(q_{s1}^{(0)}-V_{s\,1})\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(0)}-V_{s\,1})}, \quad (168)$$

so that altogether

$$\frac{\partial\Psi}{\partial n} = 4\overline{\int\left(\frac{\partial\psi_1'}{\partial n}\right)_0^n\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\infty)}-V_{s\,1})}+3\int_0^n\left[\overline{\int\frac{\partial^2\psi_1'}{\partial n^2}\,\mathrm{d}t\left(\frac{\partial^2\psi_1'}{\partial s\,\partial n}\right)_\infty^n}\right]\mathrm{d}n$$
$$+\overline{(q_{s2}^{(0)}-V_{s\,2})}+\overline{\int(q_{s1}^{(0)}-V_{s\,1})\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(0)}-V_{s\,1})}. \quad (169)$$

Suppose that the first-order motion is simple harmonic, that is, that ψ_1, $q_{s1}^{(0)}$, etc., are given by the real parts of complex quantities proportional to $e^{i\sigma t}$. Then equation (155) becomes

$$\left(i\sigma - \nu\frac{\partial^2}{\partial n^2}\right)\frac{\partial\psi_1'}{\partial n} = i\sigma(q_{s1}^{(\infty)} - V_{s\,1}), \tag{170}$$

the general solution of which is given by

$$\frac{\partial\psi_1'}{\partial n} = (q_{s1}^{(\infty)} - V_{s\,1}) + A\,e^{\alpha n} + B\,e^{-\alpha n}, \tag{171}$$

where A and B are arbitrary constants and

$$\alpha = (i\sigma/\nu)^{\frac{1}{2}} = \frac{1+i}{\delta}. \tag{172}$$

Since the solution is to remain finite in the interior of the fluid, A must vanish. The second constant B must be chosen so as to satisfy the boundary condition

$$\left(\frac{\partial\psi_1'}{\partial n}\right)_{n=0} = q_{s1}^{(0)} - V_{s\,1}. \tag{173}$$

Hence we have

$$\frac{\partial\psi_1'}{\partial n} = (q_{s1}^{(\infty)} - V_{s\,1}) + (q_{s1}^{(0)} - q_{s1}^{(\infty)})\,e^{-\alpha n}. \tag{174}$$

Thus the first-order velocity tends to its value in the interior of the fluid exponentially, but with a phase depending on n, since α is complex. A graph of the function $(e^{-\alpha n} - 1)\,e^{i\sigma t}$ for different values of t is given by Lamb (1932, p. 623) to illustrate the motion in the neighbourhood of a plane boundary when the fluid at infinity is oscillating harmonically. Equation (174) shows that in the first approximation the boundary may be regarded as plane and $q_{s1}^{(0)}$, $q_{s1}^{(\infty)}$ and $V_{s\,1}$ as independent of s as well as n. However, the fact that these velocities are not completely independent of s produces a small normal velocity relative to the boundary, given by

$$-\frac{\partial\psi_1'}{\partial s} = -n\frac{\partial}{\partial s}(q_{s1}^{(\infty)} - V_{s\,1}) + \frac{\partial}{\partial s}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\,\frac{1}{\alpha}(e^{-\alpha n} - 1). \tag{175}$$

Just beyond the boundary layer this velocity is given by

$$-\left(\frac{\partial\psi_1'}{\partial s}\right)_{n=\infty} = -\frac{1}{\alpha}\frac{\partial}{\partial s}(q_{s1}^{(0)} - q_{s1}^{(\infty)}). \tag{176}$$

In considering the second-order terms a development of the notation will be useful. If F_1 and F_2 are any two periodic quantities of the form

$$F_1 = \mathcal{R} f_1\,e^{i\sigma t}, \quad F_2 = \mathcal{R} f_2\,e^{i\sigma t}, \tag{177}$$

where f_1 and f_2 are complex and independent of t, and $\mathcal{R}$ denotes the real part, we have

$$\begin{aligned} F_1 F_2 &= \tfrac{1}{2}(f_1\,e^{i\sigma t} + f_1^*\,e^{i\sigma t}) \times \tfrac{1}{2}(f_2\,e^{i\sigma t} + f_2^*\,e^{-i\sigma t}) \\ &= \tfrac{1}{4}(f_1 f_2\,e^{2i\sigma t} + f_1^* f_2^*\,e^{-2i\sigma t} + f_1 f_2^* + f_1^* f_2), \end{aligned} \tag{178}$$

a star (*) being used to denote the conjugate complex quantity. Thus

$$\overline{F_1 F_2} = \tfrac{1}{4}(f_1 f_2^* + f_1^* f_2) = \mathcal{R}\,\tfrac{1}{2} f_1 f_2^* = \mathcal{R}\,\tfrac{1}{2} f_1^* f_2. \tag{179}$$

If the symbol $\mathscr{R}$ is omitted in (177) and (179) we may write

$$\overline{F_1 F_2} = \tfrac{1}{2} F_1 F_2^* = \tfrac{1}{2} F_1^* F_2, \tag{180}$$

it being understood that the real part only is to be taken. Any group of terms in a product may therefore be replaced as a whole by the conjugate complex group of terms.

From equation (174) we have

$$\left.\begin{aligned} \left(\frac{\partial \psi_1'}{\partial n}\right)_0^n &= (q_{s1}^{(0)} - q_{s1}^{(\infty)})\,(\mathrm{e}^{-\alpha n} - 1), \\ \frac{\partial^2 \psi_1'}{\partial n^2} &= -(q_{s1}^{(0)} - q_{s1}^{(\infty)})\,\alpha\,\mathrm{e}^{-\alpha n}, \\ \left(\frac{\partial^2 \psi_1'}{\partial s\,\partial n}\right)_\infty^n &= \frac{\partial}{\partial s}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\,\mathrm{e}^{-\alpha n}. \end{aligned}\right\} \tag{181}$$

On substituting these results in (169) and integrating the second term using (172) we find

$$\frac{\partial \Psi}{\partial n} = \frac{2}{\mathrm{i}\sigma}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\frac{\partial}{\partial s}(q_{s1}^{(\infty)*} - V_{s1}^*)\,(\mathrm{e}^{-\alpha n} - 1) + \frac{3(1+\mathrm{i})}{4\mathrm{i}\sigma}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\frac{\partial}{\partial s}(q_{s1}^{(0)*} - q_{s1}^{(\infty)*})\,(\mathrm{e}^{-(\alpha+\alpha^*)n} - 1)$$
$$+ \overline{(q_{s2}^{(0)} - V_{s2})} + \frac{1}{2\mathrm{i}\sigma}(q_{s1}^{(0)} - V_{s1})\frac{\partial}{\partial s}(q_{s1}^{(0)*} - V_{s1}^*). \tag{182}$$

Just beyond the boundary layer we have

$$\left(\frac{\partial \Psi}{\partial n}\right)_{n=\infty} = -\frac{2}{\mathrm{i}\sigma}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\frac{\partial}{\partial s}(q_{s1}^{(\infty)*} - V_{s1}^*) - \frac{3(1+\mathrm{i})}{4\mathrm{i}\sigma}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\frac{\partial}{\partial s}(q_{s1}^{(0)*} - q_{s1}^{(\infty)*})$$
$$+ \overline{(q_{s2}^{(0)} - V_{s2})} + \frac{1}{2\mathrm{i}\sigma}(q_{s1}^{(0)} - V_{s1})\frac{\partial}{\partial s}(q_{s1}^* - V_{s1}^*). \tag{183}$$

Thus the tangential component of the mass-transport velocity is determined by the tangential velocity $q_s^{(0)}$ at the boundary, the tangential velocity $q_{s1}^{(\infty)}$ just beyond the boundary layer, and the velocity V_s, which depends partly on the movement of the boundary and partly on the choice of origin. When there is no stretching of the boundary we may take

$$V_s = q_s^{(0)}. \tag{184}$$

Equations (182) and (183) then become

$$\frac{\partial \Psi}{\partial n} = \frac{1}{4\mathrm{i}\sigma}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\frac{\partial}{\partial s}(q_{s1}^{(0)*} - q_{s1}^{(\infty)*})\,[8(1 - \mathrm{e}^{-\alpha n}) + 3(1+\mathrm{i})\,(\mathrm{e}^{-(\alpha+\alpha^*)n} - 1)] \tag{185}$$

and

$$\left(\frac{\partial \Psi}{\partial n}\right)_{n=\infty} = \frac{5 - 3\mathrm{i}}{4\mathrm{i}\sigma}(q_{s1}^{(0)} - q_{s1}^{(\infty)})\frac{\partial}{\partial s}(q_{s1}^{(0)*} - q_{s1}^{(\infty)*}), \tag{186}$$

respectively. In particular when the boundary is stationary we have

$$q_s^{(0)} = 0, \tag{187}$$

and so

$$\frac{\partial \Psi}{\partial n} = \frac{1}{4\mathrm{i}\sigma} q_{s1}^{(\infty)} \frac{\partial q_{s1}^{(\infty)*}}{\partial s}\,[8(1 - \mathrm{e}^{-\alpha n}) + 3(1+\mathrm{i})\,(\mathrm{e}^{-(\alpha+\alpha^*)n} - 1)] \tag{188}$$

and

$$\left(\frac{\partial \Psi}{\partial n}\right)_{n=\infty} = \frac{5 - 3\mathrm{i}}{4\mathrm{i}\sigma} q_{s1}^{(\infty)} \frac{\partial q_{s1}^{(\infty)*}}{\partial s}. \tag{189}$$

At the boundary itself the normal component of mass-transport velocity vanishes and so Ψ must be constant. In general the normal component of the mass-transport velocity can be found from the equation

$$\frac{\partial \Psi}{\partial s} = \frac{\partial}{\partial s}\int_0^n \frac{\partial \Psi}{\partial n}\,\mathrm{d}n. \tag{190}$$

For example, when the boundary is stationary we have from (188)

$$\frac{\partial \Psi}{\partial s} = -\frac{1}{4\mathrm{i}\sigma}\left(\frac{\partial q_{s1}^{(\infty)}}{\partial s}\frac{\partial q_{s1}^{(\infty)*}}{\partial s} + q_{s1}^{(\infty)}\frac{\partial^2 q_{s1}^{(\infty)*}}{\partial s^2}\right)\frac{1}{\alpha}[8(1-\alpha n - \mathrm{e}^{-\alpha n}) + 3\mathrm{i}(\mathrm{e}^{-(\alpha+\alpha^*)n} + (\alpha+\alpha^*)\,n - 1)], \tag{191}$$

and so

$$\left(\frac{\partial \Psi}{\partial s}\right)_{n=\infty} = -\frac{1}{4\mathrm{i}\sigma}\left(\frac{\partial q_{s1}^{(\infty)}}{\partial s}\frac{\partial q_{s1}^{(\infty)*}}{\partial s} + q_{s1}^{(\infty)}\frac{\partial^2 q_{s1}^{(\infty)*}}{\partial s^2}\right)\frac{1}{\alpha}[(8-3\mathrm{i}) - (8-3\mathrm{i})\,\alpha n + 3\mathrm{i}\alpha^* n]. \tag{192}$$

8. The stresses are prescribed at the boundary

Let $p_{ns}^{(0)}$ denote the tangential stress at the boundary, which is assumed to be of order $\rho\nu a\sigma/l$. We have then

$$\left(\frac{\partial \psi}{\partial s}\right)_{n=0} = -V_n, \qquad \left[\left(\frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial s^2} + \kappa\frac{\partial}{\partial n}\right)\psi\right]_{n=0} = \frac{1}{\rho\nu}p_{ns}^{(0)}. \tag{193}$$

If we assume

$$\frac{\partial^2 \psi}{\partial n^2} = O(1) \tag{194}$$

(in powers of δ/l), it follows that $\partial\psi/\partial s$ and $\partial\psi/\partial n$ are constant, to this order, throughout the boundary layer. Thus

$$\frac{\partial \psi}{\partial s} = -V_n = -q_n^{(0)}, \qquad \frac{\partial \psi'}{\partial s} = O(\delta/l), \tag{195}$$

$$\frac{\partial \psi}{\partial n} = q_s^{(0)}, \qquad \frac{\partial \psi'}{\partial n} = q_s^{(0)} - V_s. \tag{196}$$

The second of the two boundary conditions (193) may therefore be written

$$\left(\frac{\partial^2 \psi}{\partial n^2}\right)_{n=0} = \frac{1}{\rho\nu}p_{ns}^{(0)} - \left(\frac{\partial V_n}{\partial s} + \kappa q_s^{(0)}\right), \tag{197}$$

or alternatively, since, when $n = 0$,

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial n^2} - \kappa\frac{\partial}{\partial n}\right), \tag{198}$$

we have

$$(\nabla^2\psi)_{n=0} = \frac{1}{\rho\nu}p_{ns}^{(0)} - 2\left(\frac{\partial V_n}{\partial s} + \kappa q_s^{(0)}\right). \tag{199}$$

But the value of $\partial^2\psi/\partial n^2$ just beyond the boundary layer in general differs from that given by (197). Therefore we must have

$$\frac{\partial^3 \psi}{\partial n^3} = O(\delta/l)^{-1}, \qquad \frac{\partial^4 \psi}{\partial n^4} = O(\delta/l)^{-2}, \tag{200}$$

and hence also

$$\frac{\partial^2 \psi'}{\partial n^2} = O(1), \qquad \frac{\partial^3 \psi'}{\partial n^3} = O(\delta/l)^{-1}, \qquad \frac{\partial^4 \psi'}{\partial n^4} = O(\delta/l)^{-2}, \tag{201}$$

each further differentiation with respect to n raising the order of magnitude by $(\delta/l)^{-1}$. It should be noticed, however, that there is a break in the sequence, since both $\partial\psi'/\partial n$ and $\partial^2\psi'/\partial n^2$ are $O(1)$. This introduces a significant difference between the present case and that considered in §7.

The equation of motion (117) now becomes

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi'}{\partial n}\frac{\partial}{\partial s}-\frac{\partial\psi'}{\partial s}\frac{\partial}{\partial n}-\nu\frac{\partial^2}{\partial n^2}\right)\nabla^2\psi=0. \tag{202}$$

This may be compared with equation (152), where the corresponding terms are of a higher order of magnitude. It is not possible in the present case to replace $\nabla^2\psi$ by $\partial^2\psi/\partial n^2$ or by $\partial^2\psi'/\partial n^2$. In the first approximation we now have

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial n^2}\right)\nabla^2\psi_1=0, \tag{203}$$

and so

$$\nabla^2\psi_1=\nu\int\frac{\partial^2}{\partial n^2}\nabla^2\psi_1\,\mathrm{d}t=\nu\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t. \tag{204}$$

On taking mean values in equation (202) we find

$$\overline{\left(\frac{\partial\psi_1'}{\partial n}\frac{\partial}{\partial s}-\frac{\partial\psi_1'}{\partial s}\frac{\partial}{\partial n}\right)\nabla^2\psi_1}=\nu\frac{\partial^2}{\partial n^2}\nabla^2\overline{\psi}_2=\nu\frac{\partial^4\overline{\psi_2'}}{\partial n^4}, \tag{205}$$

so that, by (204),

$$\overline{\left(\frac{\partial\psi_1'}{\partial n}\frac{\partial}{\partial s}-\frac{\partial\psi_1'}{\partial s}\frac{\partial}{\partial n}\right)\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t}-\frac{\partial^4\overline{\psi_2'}}{\partial n^4}=0, \tag{206}$$

as in equation (159). The mass-transport velocity is constant throughout the boundary layer; for from (139)

$$\left.\begin{aligned}\dot{s}&=\frac{\partial\psi'}{\partial n}=q_s^{(0)}-V_s=O(1), & \dot{n}&=-\frac{\partial\psi'}{\partial s}=O(\delta/l),\\ \frac{\partial\dot{s}}{\partial n}&=\frac{\partial^2\psi'}{\partial n^2}+\kappa\frac{\partial\psi'}{\partial n}+\Omega=O(1), & \frac{\partial\dot{n}}{\partial s}&=-\frac{\partial^2\psi'}{\partial s\,\partial n}=O(1),\end{aligned}\right\} \tag{207}$$

and so from (140)

$$\frac{\partial\Psi}{\partial n}=\overline{(q_{s2}^{(0)}-V_{s\,2})}+\overline{\int(q_{s1}^{(0)}-V_{s\,1})\,\mathrm{d}t\,(q_{s1}^{(0)}-V_{s\,1})}. \tag{208}$$

However the velocity gradient $\partial^2\Psi/\partial n^2$ is not constant. We have from (142), after differentiating four times, using Leibniz's theorem, and retaining only the terms of highest order,

$$\frac{\partial^4\Psi}{\partial n^4}=\frac{\partial^4\overline{\psi_2'}}{\partial n^4}+\overline{\int\frac{\partial^5\psi_1'}{\partial n^5}\,\mathrm{d}t\frac{\partial\psi_1'}{\partial s}}+4\overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n}}+\overline{\int\frac{\partial\psi_1'}{\partial n}\,\mathrm{d}t\frac{\partial^5\psi_1'}{\partial s\,\partial n^4}}, \tag{209}$$

which may be compared with equation (161). Thus, from (206),

$$\frac{\partial^4\Psi}{\partial n^4}=4\overline{\int\frac{\partial^4\psi_1'}{\partial n^4}\,\mathrm{d}t\frac{\partial^2\psi_1'}{\partial s\,\partial n}} \tag{210}$$

$$=4\overline{\int\frac{\partial^2}{\partial n^2}\nabla^2\psi_1\,\mathrm{d}t\frac{\partial}{\partial s}(q_s^{(0)}-V_{s\,1})}. \tag{211}$$

On integrating from $n=\infty$ we have

$$\frac{\partial^3\Psi}{\partial n^3}=4\overline{\int\frac{\partial}{\partial n}\nabla^2\psi_1\,\mathrm{d}t\frac{\partial}{\partial s}(q_s^{(0)}-V_{s\,1})}, \tag{212}$$

and on integrating from $n = 0$

$$\left(\frac{\partial^2\Psi}{\partial n^2}\right)_0^n = 4\overline{\int (\nabla^2\psi_1)_0^n \,\mathrm{d}t \frac{\partial}{\partial s}(q_{s1}^{(0)} - V_{s1})}. \tag{213}$$

To obtain a boundary condition for Ψ when $n = 0$ we have from (140) and (141), after differentiating with respect to n,

$$\frac{\partial}{\partial n}\left(\frac{1}{\eta_0}\frac{\partial\Psi}{\partial n}\right) = \frac{\partial \bar{\dot{s}}_2}{\partial n} + \overline{\int \frac{\partial \dot{s}_1}{\partial n}\mathrm{d}t \frac{\partial \dot{s}_1}{\partial s}} + \overline{\int \dot{s}_1 \,\mathrm{d}t \frac{\partial^2 \dot{s}_1}{\partial s \partial n}} + \overline{\int \frac{\partial \dot{n}_1}{\partial n}\mathrm{d}t \frac{\partial \dot{s}_1}{\partial n}} + \overline{\int \dot{n}_1 \,\mathrm{d}t \frac{\partial^2 \dot{s}_1}{\partial n^2}}, \tag{214}$$

or, since $\dot{n}_1$ vanishes and $\partial\dot{n}_1/\partial n$ equals $-\partial\dot{s}_1/\partial s$,

$$\left(\frac{\partial^2\Psi}{\partial n^2} + \kappa_0 \frac{\partial\Psi}{\partial n}\right)_{n=0} = \frac{\partial \bar{\dot{s}}_2}{\partial n} + 2\overline{\int \frac{\partial \dot{s}_1}{\partial n}\mathrm{d}t \frac{\partial \dot{s}_1}{\partial s}} + \overline{\int \dot{s}_1 \,\mathrm{d}t \frac{\partial^2 \dot{s}_1}{\partial s \partial n}}. \tag{215}$$

But from (207)

$$\frac{\partial \dot{s}}{\partial n} = \frac{\partial^2\psi}{\partial n^2} + \kappa(q_s^{(0)} - V_s) + \Omega, \tag{216}$$

so that from (197) and (132)

$$\left(\frac{\partial \dot{s}}{\partial n}\right)_{n=0} = \frac{1}{\rho\nu} p_{ns}^{(0)}. \tag{217}$$

Hence

$$\left(\frac{\partial^2\Psi}{\partial n^2}\right)_{n=0} + \kappa_0 \frac{\partial\Psi}{\partial n} = \frac{1}{\rho\nu}\left[\overline{p_{ns2}^{(0)}} + 2\overline{\int p_{ns1}^{(0)}\,\mathrm{d}t \frac{\partial}{\partial s}(q_{s1}^{(0)} - V_{s1})} + \overline{\int (q_{s1}^{(0)} - V_{s1})\,\mathrm{d}t \frac{\partial p_{ns1}^{(0)}}{\partial s}}\right]. \tag{218}$$

Thus altogether

$$\begin{aligned}\frac{\partial^2\Psi}{\partial n^2} + \kappa_0 \frac{\partial\Psi}{\partial n} = {} & 4\overline{\int (\nabla^2\psi_1)_0^n \,\mathrm{d}t \frac{\partial}{\partial s}(q_{s1}^{(0)} - V_{s1})} \\ & + \frac{1}{\rho\nu}\left[\overline{p_{ns2}^{(0)}} + 2\overline{\int p_{ns1}^{(0)}\,\mathrm{d}t \frac{\partial}{\partial s}(q_{s1}^{(0)} - V_{s1})} + \overline{\int (q_{s1}^{(0)} - V_{s1})\,\mathrm{d}t \frac{\partial p_{ns1}^{(0)}}{\partial s}}\right].\end{aligned} \tag{219}$$

From (219) we have, on replacing V_{n1} by $q_{n1}^{(0)}$,

$$(\nabla^2\psi_1)_{n=0} = \frac{1}{\rho\nu} p_{ns1}^{(0)} - 2\left(\frac{\partial q_{n1}^{(0)}}{\partial s} + \kappa_0 q_{s1}^{(0)}\right), \tag{220}$$

so that (219) can be written

$$\begin{aligned}\frac{\partial^2\Psi}{\partial n^2} + \kappa_0 \frac{\partial\Psi}{\partial n} = {} & 4\overline{\int \nabla^2\psi_1 \,\mathrm{d}t \frac{\partial}{\partial s}(q_{s1}^{(0)} - V_{s1})} + 8\overline{\int \left(\frac{\partial q_{n1}^{(0)}}{\partial s} + \kappa_0 q_{s1}^{(0)}\right)\mathrm{d}t \frac{\partial}{\partial s}(q_{s1}^{(0)} - V_{s1})} \\ & + \frac{1}{\rho\nu}\left[\overline{p_{ns2}^{(0)}} - 2\overline{\int p_{ns1}^{(0)}\,\mathrm{d}t \frac{\partial}{\partial s}(q_{s1}^{(0)} - V_{s1})} + \overline{\int (q_{s1}^{(0)} - V_{s1})\,\mathrm{d}t \frac{\partial p_{ns1}^{(0)}}{\partial s}}\right].\end{aligned} \tag{221}$$

Also Ψ is to be constant along the boundary.

When the first-order motion is simple harmonic, equation (203) becomes

$$\left(\mathrm{i}\sigma - \nu\frac{\partial^2}{\partial n^2}\right)\nabla^2\psi_1 = 0. \tag{222}$$

The only solution of this equation which is finite for large n and which satisfies the boundary condition (220) is

$$\nabla^2\psi_1 = \left[\frac{1}{\rho\nu} p_{ns1}^{(0)} - 2\left(\frac{\partial q_{n1}^{(0)}}{\partial s} + \kappa_0 q_{s1}^{(0)}\right)\right]\mathrm{e}^{-\alpha n}. \tag{223}$$

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Thus from (219)

$$\frac{\partial^2\Psi'}{\partial n^2}+\kappa_0\frac{\partial\Psi'}{\partial n}=\frac{2}{\mathrm{i}\sigma}\left[\frac{1}{\rho\nu}-2\left(\frac{\partial q_{n1}^{(0)}}{\partial s}+\kappa_0 q_{s1}^{(0)}\right)\right]\frac{\partial}{\partial s}\left(q_{s1}^{(0)*}-V_{s1}^*\right)\left(\mathrm{e}^{-\alpha n}-1\right)$$
$$+\frac{1}{\rho\nu}\left[\overline{p_{ns2}^{(0)}}+\frac{1}{\mathrm{i}\sigma}p_{ns1}\frac{\partial}{\partial s}\left(q_{s1}^{(0)*}-V_{s1}^*\right)+\frac{1}{2\mathrm{i}\sigma}\left(q_{s1}^{(0)}-V_{s1}\right)\frac{\partial p_{ns1}^{(0)*}}{\partial s}\right], \tag{224}$$

and just beyond the boundary layer we have from (221)

$$\left(\frac{\partial^2\Psi'}{\partial n^2}+\kappa_0\frac{\partial\Psi'}{\partial n}\right)_{n=\infty}=\frac{4}{\mathrm{i}\sigma}\left(\frac{\partial q_{n1}^{(0)}}{\partial s}+\kappa_0 q_{s1}^{(0)}\right)\frac{\partial}{\partial s}\left(q_{s1}^{(0)*}-V_{s1}^*\right)$$
$$+\frac{1}{\rho\nu}\left[\overline{p_{ns2}^{(0)}}-\frac{1}{\mathrm{i}\sigma}p_{ns1}\frac{\partial}{\partial s}\left(q_{s1}^{(0)*}-V_{s1}^*\right)+\frac{1}{2\mathrm{i}\sigma}\left(q_{s1}^{(0)}-V_{s1}\right)\frac{\partial p_{ns1}^{(0)*}}{\partial s}\right]. \tag{225}$$

At a free surface $p_{ns}^{(0)}$ vanishes, making the last group of terms in (224) and (225) zero. If also $\kappa_0=0$, then the co-ordinates may be chosen so that V_{s1} vanishes (see § 6). We have then

$$\frac{\partial^2\Psi'}{\partial n^2}=\frac{4}{\mathrm{i}\sigma}\frac{\partial q_{n1}^{(0)}}{\partial s}\frac{\partial q_{s1}^{(0)*}}{\partial s}\left(1-\mathrm{e}^{-\alpha n}\right), \tag{226}$$

and just beyond the boundary layer

$$\left(\frac{\partial^2\Psi'}{\partial n^2}\right)_{n=\infty}=\frac{4}{\mathrm{i}\sigma}\frac{\partial q_{n1}^{(0)}}{\partial s}\frac{\partial q_{s1}^*}{\partial s}. \tag{227}$$

Thus, in the present case, it is the normal gradient of the mass transport which is determined throughout the boundary layer.

9. Determination of Ψ' in the interior

Suppose that it is desired to find a periodic motion satisfying, at the boundaries, one of two types of condition: either the normal and the tangential velocities are prescribed to be equal to $q_n^{(0)}$ and $q_s^{(0)}$ respectively or else the normal and tangential stresses are to be equal to $p_{nn}^{(0)}$ and $p_{ns}^{(0)}$ respectively. Suppose also that a perfect-fluid solution ψ_a exists, satisfying Laplace's equation in the interior of the fluid, having the normal velocity $q_n^{(0)}$ at the first type of boundary, and having a value of p equal to $-p_{nn}^{(0)}$ at the second type of boundary. (ψ_a will not of course satisfy the other two boundary conditions, in general.) Let $\epsilon\psi_{a1}$ be the first approximation to ψ_a, in powers of ϵ; ψ_{a1} is to be considered as being referred to co-ordinates normal and tangential to the boundaries. Since ψ_{a1} satisfies Laplace's equation, it satisfies also the equations of viscous motion in the interior of the fluid. Also, since p, apart from the hydrostatic pressure, is a function of the velocities of the order of $\rho a\sigma^2 l$ (see equation (122)) and since, from equation (94),

$$p_{nn}=-p+O(\rho\nu a\sigma/l) \tag{228}$$

it follows that $\epsilon\psi_{a1}$ gives the prescribed value of the normal stress p_{nn}, with relative errors of the order of $(\delta/l)^2$. Further, from §§ 7 and 8 we see that by adding to ψ_{a1} functions, say ψ_{b1}, which vanish exponentially inwards from the boundaries, the conditions of prescribed tangential stress or velocity may be satisfied (to order ϵ). The functions $\epsilon\psi_{b1}$ produce also additional stresses and velocities normal to the boundaries. But, considered as functions of the velocities, these are at most of order δ/l relative to the corresponding functions for $\epsilon\psi_{a1}$. Hence $\epsilon(\psi_{a1}+\psi_{b1})$ satisfies all the prescribed boundary conditions for $\epsilon\psi_1$, with neglect only of δ/l. In the interior of the fluid $\epsilon\psi_1$ tends exponentially to $\epsilon\psi_{a1}$. Hence the velocities $q_{s1}^{(\infty)}$, $q_{s1}^{(0)}$, V_{s1} and V_{n1} may be calculated by the ordinary theory of perfect fluids, and will be correct to order δ/l in the interior. The effect of surface tension, which enters only into the normal stress, may be taken into account by calculating its effect on ψ_{a1} in the usual way.

Thus the theory of perfect fluids can be expected to describe the motion in the interior of the fluid successfully to order ϵ. But to order ϵ^2 this is not so. From § 7 we see that when the normal and tangential velocities at the boundary are given, the mass-transport velocity $\partial\Psi/\partial n$ just beyond the boundary layer is well determined, and not arbitrary as in the theory of a perfect fluid. To the present order of approximation the velocity is independent of the viscosity, and, as ν tends to zero, it tends to a value different from that at the boundaries. This phenomenon was noticed by Schlichting (1932) and Rayleigh (1883) in special cases. Similarly, from § 8, when the tangential stress is prescribed at the boundary the normal gradient of the mass-transport velocity, or rather $\left(\dfrac{\partial^2\Psi}{\partial n^2}+\kappa_0\dfrac{\partial\Psi}{\partial n}\right)$, is determined just beyond the boundary layer.

To determine Ψ in the interior of the fluid, suppose that the first-order solution ψ_{a1} is found by the classical theory; $q_{s1}^{(0)}, q_{n1}^{(0)}, q_{s1}^{(\infty)}$, etc., are then known. At the first type of boundary we have

$$\frac{\partial\Psi}{\partial s}=0, \tag{229}$$

and just beyond the boundary layer (if the velocities are expressed as the real parts of complex quantities)

$$\frac{\partial\Psi}{\partial n}=4\overline{\int(q_{s1}^{(\infty)}-q_{s1}^{(0)})\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\infty)}-V_{s1})}-\frac{3(1+\mathrm{i})}{2}\overline{\int(q_{s1}^{(\infty)}-q_{s1}^{(0)})\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(\alpha)}-q_{s1}^{(0)})}$$
$$+\overline{(q_{s2}^{(0)}-V_{s2})}+\overline{\int(q_{s1}^{(0)}-V_{s1})\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(0)}-V_{s1})} \tag{230}$$

by (183). Since the boundary layer is only of thickness δ, the second condition may be supposed to be satisfied at the boundary itself, or at the mean boundary, for this will not affect the value of Ψ in the interior to the present approximation. Similarly, at the second type of boundary we have

$$\frac{\partial\Psi}{\partial s}=0, \tag{231}$$

and from (221)

$$\frac{\partial^2\Psi}{\partial n^2}+\kappa_0\frac{\partial\Psi}{\partial n}=8\overline{\int\left(\frac{\partial q_{n1}^{(0)}}{\partial s}+\kappa_0 q_{s1}^{(0)}\right)\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(0)}-V_{s1})}$$
$$+\frac{1}{\rho\nu}\left[\overline{p_{ns2}^{(0)}}-2\overline{\int p_{ns1}^{(0)}\,\mathrm{d}t\frac{\partial}{\partial s}(q_{s1}^{(0)}-V_{s1})}+\overline{\int(q_{s1}^{(0)}-V_{s1})\,\mathrm{d}t\frac{\partial p_{ns1}^{(0)}}{\partial s}}\right]. \tag{232}$$

In the interior of the fluid the field equation may be taken to be either (68) or (86), according as $a^2\ll\delta^2$ or $a^2\gg\delta^2$. Solutions of these equations may be called conduction solutions and convection solutions respectively, corresponding to the names for the equations suggested in § 4.

Since the conduction equation is of the fourth order we may expect that a conduction solution satisfying all the boundary conditions exists in general. But since the convection equation is only of the second order, a convection solution can be expected to exist only in special cases.

Let us consider more closely the conditions to be satisfied by the convection solution near a free surface, say $z = 0$. Setting $p_{ns}^{(0)} = \kappa_0 = V_{s1} = 0$ in equation (232), we have

$$\frac{\partial^2 \Psi}{\partial n^2} = 8 \overline{\int \frac{\partial q_{n1}^{(0)}}{\partial s} \mathrm{d}t \frac{\partial q_{s1}^{(0)}}{\partial s}}, \tag{233}$$

or, on replacing (s, n) by (x, z),

$$\left(\frac{\partial^2 \Psi}{\partial z^2}\right)_{z=0} = -8 \overline{\int \frac{\partial^2 \psi_{a1}}{\partial x^2} \mathrm{d}t \frac{\partial^2 \psi_{a1}}{\partial x \partial z}}. \tag{234}$$

Now since Ψ is constant when $z = 0$ we have from the convection equation (86)

$$\left(\frac{\partial^2 \Psi}{\partial z^2}\right)_{z=0} = \text{constant} - 4 \overline{\int \frac{\partial^2 \psi_{a1}}{\partial x^2} \mathrm{d}t \frac{\partial^2 \psi_{a1}}{\partial x \partial z}}. \tag{235}$$

From (234) and (235) it follows that a necessary condition is

$$\overline{\int \frac{\partial^2 \psi_{a1}}{\partial x^2} \mathrm{d}t \frac{\partial^2 \psi_{a1}}{\partial x \partial z}} = \text{constant}, \tag{236}$$

or

$$\overline{\int \frac{\partial^2 \psi_{a1}}{\partial z^2} \mathrm{d}t \frac{\partial^2 \psi_{a1}}{\partial x \partial z}} = \text{constant}, \tag{237}$$

since ψ_{a1} satisfies Laplace's equation. This is equivalent to

$$\overline{\int \frac{\partial^2 \psi_{a1}}{\partial x \partial z^2} \mathrm{d}t \frac{\partial^2 \psi_{a1}}{\partial x \partial z}} + \overline{\frac{\partial^2 \psi_{a1}}{\partial z^2} \int \frac{\partial^3 \psi_{a1}}{\partial z^3} \mathrm{d}t} = 0 \tag{238}$$

(on differentiating with respect to x and using the property of the periodicity, equation (34)). Now the condition of constant normal pressure at the free surface gives, for the perfect-fluid solution,

$$\frac{\partial^2 \phi_{a1}}{\partial t^2} - g \frac{\partial \phi_{a1}}{\partial z} = 0, \tag{239}$$

where ϕ_{a1} is the velocity potential corresponding to ψ_{a1} (Stokes 1847). On differentiating with respect to x and replacing $\partial \phi_{a1}/\partial x$ by $-\partial \psi_{a1}/\partial z$ we have

$$\frac{\partial^3 \psi_{a1}}{\partial z \partial t^2} - g \frac{\partial \psi_{a1}}{\partial z^2} = 0. \tag{240}$$

Thus the left-hand side of (238) may be written

$$\frac{1}{g}\left[\overline{\frac{\partial^3 \psi_{a1}}{\partial x \partial z \partial t} \frac{\partial^2 \psi_{a1}}{\partial x \partial z}} + \overline{\frac{\partial^2 \psi_{a1}}{\partial z^2} \frac{\partial^3 \psi_{a1}}{\partial z^2 \partial t}}\right]. \tag{241}$$

Each term vanishes, by the periodicity; thus the necessary condition is satisfied. The proof can be extended to the case when the surface tension is taken into account. Hence, if both normal and tangential stresses at the surface vanish, it may be possible to satisfy the condition (232); but if the stresses do not vanish a convection solution cannot in general be found.

PART III. WAVES IN WATER OF UNIFORM DEPTH

10. Introduction

In parts I and II a general method was described for finding the mass-transport velocity in any oscillatory motion of small amplitude, given the first-order motion for a perfect fluid. In this part the method will be applied to the case of waves in water of uniform depth.

As shown in part I, the motion in the interior of the fluid has a different character when the ratio a^2/δ^2 is small, and when it is large, compared with unity (a denotes the amplitude of the first-order oscillation, and $\delta = (2\nu/\sigma)^{\frac{1}{2}}$, where ν is the viscosity and $2\pi/\sigma$ is the period). In the first case the vorticity is diffused throughout the fluid by viscous conduction, and in the second case by convection with the mass-transport velocity. There are two different field equations for the two cases (equations (68) and (84)). The mass transport near the boundaries, however, does not depend critically on the ratio a/δ, but is determined by the first-order motion and the local boundary conditions. The thickness of the boundary layer is of order δ. Just beyond this layer either the mass-transport velocity $\partial\Psi/\partial n$ itself or its normal gradient $\partial^2\Psi/\partial n^2$ takes a certain definite value, depending on whether the boundary is fixed or 'free'; and, by combining the known values of $\partial\Psi/\partial n$ or $\partial^2\Psi/\partial n^2$ just beyond the boundaries with the approximate field equations for the stream function Ψ in the interior of the fluid, a 'conduction solution' or a 'convection solution' may be obtained.

In order to treat the progressive and the standing wave together we shall consider a motion which, in the first approximation, consists of two waves of the same period and wave-length travelling in opposite directions; that is, we suppose that the equation of the free surface is

$$z = a_1 \cos(kx-\sigma t)+a_2\cos(kx+\sigma t) \tag{242}$$

in the usual notation; or, if the real part only is taken,

$$z = (a_1\,\mathrm{e}^{-\mathrm{i}kx}+a_2\,\mathrm{e}^{\mathrm{i}kx})\,\mathrm{e}^{\mathrm{i}\sigma t}. \tag{243}$$

The corresponding stream function is given by

$$\epsilon\psi_{a1} = -\frac{\sigma}{k}\frac{\sinh k(z-h)}{\sinh kh}(a_1\,\mathrm{e}^{-\mathrm{i}kx}-a_2\,\mathrm{e}^{\mathrm{i}kx})\,\mathrm{e}^{\mathrm{i}\sigma t} \tag{244}$$

(Stokes 1847). The condition of constant pressure at the free surface gives

$$\sigma^2 = gk\tanh kh. \tag{245}$$

To obtain a single progressive wave of amplitude a travelling in the direction of x increasing we shall write

$$a_1 = a, \qquad a_2 = 0. \tag{246}$$

To obtain a standing wave of amplitude $2a$ we shall write

$$a_1 = a_2 = a. \tag{247}$$

We shall first evaluate the motion in the boundary layers, and then proceed to consider the motion in the interior of the fluid. Finally, the results will be compared with some observations of mass-transport velocities.

11. The boundary layer at the bottom

In equation (188) we write

$$s = -x, \quad n = h - z, \tag{248}$$

and

$$\epsilon q_{s1}^{(\infty)} = \frac{\sigma}{\sinh kh}(a_1\,\mathrm{e}^{-kx} - a_2\,\mathrm{e}^{ikx})\,\mathrm{e}^{i\sigma t}. \tag{249}$$

This gives

$$\epsilon^2 \frac{\partial \Psi}{\partial z} = \frac{\sigma k}{4\sinh^2 kh}(a_1^2 - a_2^2 - 2a_1 a_2 \sin 2kx)\,[8(1-\mathrm{e}^{-\alpha(h-z)}) - 3(1+\mathrm{i})\,(1-\mathrm{e}^{-(\alpha+\alpha^*)(h-z)})]. \tag{250}$$

Now

$$\alpha(h-z) = (1+\mathrm{i})\frac{h-z}{\delta}. \tag{251}$$

Thus, retaining only the real part, we have

$$\epsilon^2 \frac{\partial \Psi}{\partial z} = \frac{\sigma k}{4\sinh^2 kh}\left[(a_1^2 - a_2^2)\,f^{(p)}\left(\frac{h-z}{\delta}\right) + 2a_1 a_2 \sin 2kx\, f^{(s)}\left(\frac{h-z}{\delta}\right)\right], \tag{252}$$

where

$$\left.\begin{aligned} f^{(p)}(\mu) &= 5 - 8\,\mathrm{e}^{-\mu}\cos\mu + 3\,\mathrm{e}^{-2\mu}, \\ f^{(s)}(\mu) &= -3 + 8\,\mathrm{e}^{-\mu}\sin\mu + 3\,\mathrm{e}^{-2\mu}. \end{aligned}\right\} \tag{253}$$

(a) *The progressive wave*

Assuming (246) we have

$$\epsilon^2 \frac{\partial \Psi}{\partial z} = \frac{a^2 \sigma k}{4\sinh^2 kh} f^{(p)}\left(\frac{h-z}{\delta}\right). \tag{254}$$

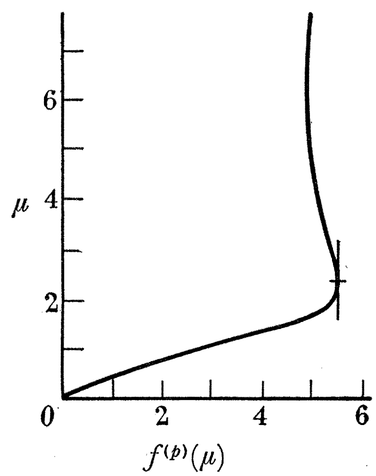

Figure 4. Graph of $f^{(p)}(\mu)$, representing the profile of the mass-transport velocity in the boundary layer at the bottom, in a progressive wave.

$f^{(p)}(\mu)$, which represents the typical velocity profile near the bottom for the progressive wave, is plotted against μ in figure 4. It will be seen that $f^{(p)}(\mu)$ is always positive and, when μ tends to infinity, tends to the value 5. Thus, just beyond the boundary layer,

$$\epsilon^2 \frac{\partial \Psi}{\partial z} = \frac{5a^2 \sigma k}{4\sinh^2 kh}. \tag{255}$$

Also
$$\frac{\mathrm{d}f^{(p)}}{\mathrm{d}\mu} = 8\sqrt{2}\,\mathrm{e}^{-\mu}\sin(\mu+\tfrac{1}{4}\pi) - 6\,\mathrm{e}^{-2\mu}, \tag{256}$$
so that stationary values of the velocity occur when
$$\sin(\mu+\tfrac{1}{4}\pi) = \frac{3}{4\sqrt{2}}\mathrm{e}^{-\mu}. \tag{257}$$
The lowest root is given by
$$\mu = 2{\cdot}306, \quad f^{(p)} = 5{\cdot}505, \tag{258}$$
so that the maximum value of the velocity is given by
$$\epsilon^2\left(\frac{\partial\Psi}{\partial z}\right)_{\text{max.}} = 1{\cdot}376\,\frac{a^2\sigma k}{\sinh^2 kh}. \tag{259}$$
Subsequent maxima or minima occur when
$$\mu \doteqdot (m-\tfrac{1}{4})\,\pi \quad (m = 2, 3, \ldots). \tag{260}$$

(b) *The standing wave*

Assuming (247) we have
$$\epsilon^2\frac{\partial\Psi}{\partial z} = \frac{1}{2}\frac{a^2\sigma k}{\sinh^2 kh}\sin 2kx\, f^{(s)}\left(\frac{h-z}{\delta}\right). \tag{261}$$
$f^{(s)}(\mu)$ is plotted against μ in figure 5. As μ tends to infinity $f^{(s)}$ tends to -3. Thus just beyond the boundary layer we have
$$\epsilon^2\frac{\partial\Psi}{\partial z} = -\frac{3}{2}\frac{a^2\sigma k}{\sinh^2 kh}\sin 2kx. \tag{262}$$

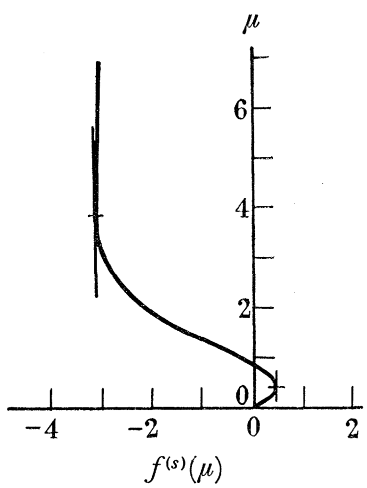

FIGURE 5. Graph of $f^{(s)}(\mu)$, representing the profile of the mass-transport velocity in the boundary layer at the bottom, in a standing wave.

However, for small values of μ, $f^{(s)}$ takes positive values. There is one zero, namely, when $\mu = 0{\cdot}93$. Since
$$\frac{\mathrm{d}f^{(s)}}{\mathrm{d}\mu} = 8\sqrt{2}\,\mathrm{e}^{-\mu}\sin(\mu+\tfrac{3}{4}\pi) - 6\,\mathrm{e}^{-2\mu}, \tag{263}$$
maxima and minima occur when
$$\sin(\mu+\tfrac{3}{4}\pi) = \frac{3\,\mathrm{e}^{-\mu}}{4\sqrt{2}}. \tag{264}$$

The first two stationary values occur when

$$\mu = 0{\cdot}49, \quad f^{(s)} = -0{\cdot}41 \tag{265}$$

and

$$\mu = 3{\cdot}94, \quad f^{(s)} = 3{\cdot}11. \tag{266}$$

Subsequent maxima or minima occur when

$$\mu \doteqdot (m - \tfrac{3}{4}\pi) \quad (m = 3, 4, \ldots). \tag{267}$$

We see from (261) that the horizontal mass-transport velocity varies as $\sin 2kx$; in the planes $x = 0, \pm\frac{1}{4}\lambda, \pm\frac{1}{2}\lambda, \ldots$, it is zero, and in the planes $x = \pm\frac{1}{8}\lambda, \pm\frac{3}{8}\lambda, \ldots$, it is a maximum. The particles very close to the bottom tend to move towards the planes of greatest horizontal first-order motion and away from the planes where the motion is purely vertical, but for larger values of $(h-z)/\delta$ the particles drift in the reverse sense. Hence there is a circulation in the boundary layer itself, in cells whose length is $\frac{1}{4}\lambda$. The vertical velocity is given by

$$\epsilon^2 \frac{\partial \Psi}{\partial z} = \frac{a^2 \delta k^2 \sigma}{\sinh^2 kh} \cos 2kx \int_0^{(h-z)/\delta} f^{(s)}(\mu)\, \mathrm{d}\mu. \tag{268}$$

This vanishes when $(h-z)/\delta = 0$ and $1{\cdot}47$.

12. The boundary layer at the free surface

Since the boundary is moving we retain at first the co-ordinates (s, n). In equation (226) we write

$$\epsilon q_{n1}^{(0)} = \mathrm{i}\sigma (a_1\, \mathrm{e}^{-\mathrm{i}ks} + a_2\, \mathrm{e}^{\mathrm{i}ks})\, \mathrm{e}^{\mathrm{i}\sigma t} \tag{269}$$

and

$$\epsilon q_{s1}^{(0)} = -\sigma \coth kh\, (a_1\, \mathrm{e}^{-\mathrm{i}ks} - a_2\, \mathrm{e}^{\mathrm{i}ks})\, \mathrm{e}^{\mathrm{i}\sigma t}, \tag{270}$$

giving

$$\epsilon^2 \frac{\partial \Psi}{\partial n^2} = -4\sigma k^2 \coth kh\, (a_1^2 - a_2^2 - 2\mathrm{i} a_1 a_2 \sin 2ks)\, (1 - \mathrm{e}^{-\alpha n}). \tag{271}$$

Thus, retaining only the real part of (271) we have

$$\epsilon^2 \frac{\partial^2 \Psi}{\partial n^2} = 4\sigma k^2 \coth kh \left[(a_1^2 - a_2^2)\, g^{(p)}\left(\frac{n}{\delta}\right) + 2a_1 a_2 \sin 2ks\, g^{(s)}\left(\frac{n}{\delta}\right) \right], \tag{272}$$

where

$$g^{(p)}(\mu) = -1 + \mathrm{e}^{-\mu} \cos \mu, \tag{273}$$

$$g^{(s)}(\mu) = -\mathrm{e}^{-\mu} \sin \mu. \tag{274}$$

(a) *The progressive wave*

In this case

$$\epsilon^2 \frac{\partial^2 \Psi}{\partial n^2} = 4a^2 \sigma k^2 \coth kh\, g^{(p)}\left(\frac{n}{\delta}\right). \tag{275}$$

As μ tends to infinity $g^{(p)}$ tends to -1. Thus, just beyond the boundary layer the velocity gradient is given by

$$\epsilon^2 \frac{\partial^2 \Psi}{\partial n^2} = -4a^2 \sigma k \coth kh, \tag{276}$$

which is twice the corresponding value for the irrotational wave (see Stokes 1847). Hence there is a vorticity given by

$$\epsilon^2 \nabla^2 \overline{\psi_2} = -2a^2 \sigma k^2 \coth kh. \tag{277}$$

From (273) we see that $g^{(p)}$ is always negative except at the free surface, where it vanishes. The stationary values of $g^{(p)}$ are given by

$$\mu = (m - \tfrac{1}{4})\, \pi \quad (m = 1, 2, \ldots). \tag{278}$$

The greatest value of the velocity gradient is given by

$$m = 1, \quad \mu = \tfrac{3}{4}\pi, \quad g^{(p)} = -1\cdot 67. \tag{279}$$

(*b*) *The standing wave*

In this case we have from (274)

$$\epsilon^2 \frac{\partial^2 \Psi'}{\partial n^2} = 8a^2 \sigma k \coth kh \sin 2ks \, g^{(s)}\left(\frac{n}{\delta}\right). \tag{280}$$

As μ tends to infinity $g^{(s)}$ tends to zero. Thus, just beyond the boundary layer

$$\epsilon^2 \frac{\partial^2 \Psi'}{\partial n^2} = 0, \tag{281}$$

as in the irrotational wave. In the boundary layer itself the velocity gradient may take both positive and negative values. The stationary values of $g^{(s)}$ are given by

$$\mu = (m - \tfrac{3}{4})\,\pi \quad (m = 1, 2, \ldots). \tag{282}$$

The greatest and least values of the velocity gradient occur when

$$\mu = \tfrac{1}{4}\pi, \quad g^{(s)} = -0\cdot 67, \tag{283}$$

and

$$\mu = \tfrac{3}{4}\pi, \quad g^{(s)} = 0\cdot 14, \tag{284}$$

respectively. The velocity gradient vanishes when

$$\mu = m\pi \quad (m = 0, 1, 2, \ldots). \tag{285}$$

13. Motion in the interior: the conduction solution

As boundary conditions for the motion in the interior we find the values of $\partial \Psi'/\partial n$ (or $\partial^2 \Psi'/\partial n^2$) just beyond the boundary layers ($n \gg \delta$), and suppose that these are to be taken at the mean boundary itself. Thus from (252) and (273) we have, so far as the motion in the interior is concerned,

$$\epsilon^2 \left(\frac{\partial \Psi'}{\partial z}\right)_{z=h} = \frac{\sigma k}{4 \sinh^2 kh}\,[5(a_1^2 - a_2^2) - 6a_1 a_2 \sin 2kx] \tag{286}$$

and

$$\epsilon^2 \left(\frac{\partial^2 \Psi'}{\partial z^2}\right)_{z=0} = -\frac{2\sigma k^2 \sinh 2kh}{\sinh^2 kh}\,(a_1^2 - a_2^2). \tag{287}$$

On each of the mean boundaries Ψ' is to be constant. The arbitrary constant in Ψ' may be chosen so as to make Ψ' vanish at the upper boundary. Thus

$$(\Psi')_{z=0} = 0 \tag{288}$$

and

$$(\Psi')_{z=h} = \text{constant}. \tag{289}$$

Also from equation (244) we have

$$\epsilon^2 \overline{\int \frac{\partial \psi_{a1}}{\partial z}\,\mathrm{d}t \frac{\partial \psi_{a1}}{\partial x}} = \frac{\sigma}{4} \frac{\sinh 2k(z-h)}{\sinh^2 kh}\,(a_1^2 - a_2^2), \tag{290}$$

so that the conduction equation for Ψ' in the interior of the fluid is, by equation (68),

$$\nabla^4 \Psi' = \nabla^4 \frac{\sigma}{4} \frac{\sinh 2k(z-h)}{\sinh^2 kh}\,(a_1^2 - a_2^2). \tag{291}$$

However, the equations (286) to (289) and (291) are not quite sufficient to determine Ψ' uniquely; one further condition is required. We may suppose that

$$(\Psi')_{z=h} = 0, \tag{292}$$

i.e. that the total horizontal flow due to the mass transport is zero. Assume a solution of the form

$$\epsilon^2\Psi' = \frac{\sigma}{4\sinh^2 kh}[(a_1^2-a_2^2)\{\sinh 2k(z-h)+Z^{(p)}(z)\}+2a_1a_2\sin 2kx\,Z^{(s)}(z)]. \tag{293}$$

Then $Z^{(p)}$ and $Z^{(s)}$ must satisfy

$$\left.\begin{aligned} &\frac{d^4Z^{(p)}}{dz^4} = 0,\\ &\left(\frac{dZ^{(p)}}{dz}\right)_{z=h} = 3k, \qquad (Z^{(p)})_{z=h} = 0,\\ &\left(\frac{d^2Z^{(p)}}{dz^2}\right)_{z=0} = -4k^2\sinh 2kh, \qquad (Z^{(p)})_{z=0} = \sinh 2kh \end{aligned}\right\} \tag{294}$$

and

$$\left.\begin{aligned} &\left(\frac{d^2}{dz^2}-4k^2\right)^2 Z^{(s)} = 0,\\ &\left(\frac{dZ^{(s)}}{dz}\right)_{z=h} = -3k, \qquad (Z^{(s)})_{z=h} = 0,\\ &\left(\frac{d^2Z^{(s)}}{dz^2}\right)_{z=0} = 0, \qquad (Z^{(s)})_{z=0} = 0. \end{aligned}\right\} \tag{295}$$

The solutions of these equations are given by

$$Z^{(p)} = \sinh 2kh+3kz+k^2h^2\sinh 2kh(z^3/h^3-2z^2/h^2+z/h)+\tfrac{1}{2}(\sinh 2kh+3kh)\,(z^3/h^3-3z/h) \tag{296}$$

and

$$Z^{(s)} = 3\,\frac{2kh\cosh 2kh\sinh 2kz-2kz\cosh 2kz\sinh 2kh}{\sinh 4kh-4kh}. \tag{297}$$

This gives the solution (293) uniquely. But if the condition (294) is relaxed an arbitrary multiple of

$$a^2\sigma(z^3/h^3-3z/h) \tag{298}$$

may be added to Ψ'. The expression (298) represents a parabolic velocity distribution which vanishes on the bottom and has zero velocity gradient at the free surface.

(a) *The progressive wave*

When $a_1 = a$, $a_2 = 0$, we have from (293) and (296)

$$\epsilon^2\frac{\partial\Psi'}{\partial z} = a^2\sigma kF^{(p)}(z/h), \tag{299}$$

where
$$F^{(p)}(\mu) = \frac{1}{4\sinh^2 kh}\Big[2\cosh 2kh(\mu-1)+3+kh\sinh 2kh(3\mu^2-4\mu+1) +3\left(\frac{\sinh 2kh}{2kh}+\frac{3}{2}\right)(\mu^2-1)\Big]. \tag{300}$$

In figure 6, $F^{(p)}$ is plotted against μ for $kh = 0{\cdot}5$, $1{\cdot}0$ and $1{\cdot}5$. It will be seen that the curve is always concave towards the right. For small values of kh the velocity is greatest near the bottom. When kh is negligible we have

$$F^{(p)}(\mu) = \frac{5}{8k^2h^2}(3\mu^2-1), \tag{301}$$

and so

$$\epsilon^2\frac{\partial\Psi}{\partial z} = \frac{5b^2\sigma k}{8}(3z^2/h^2-1), \tag{302}$$

where b is the amplitude of the horizontal motion:

$$b = a\coth kh = a/kh. \tag{303}$$

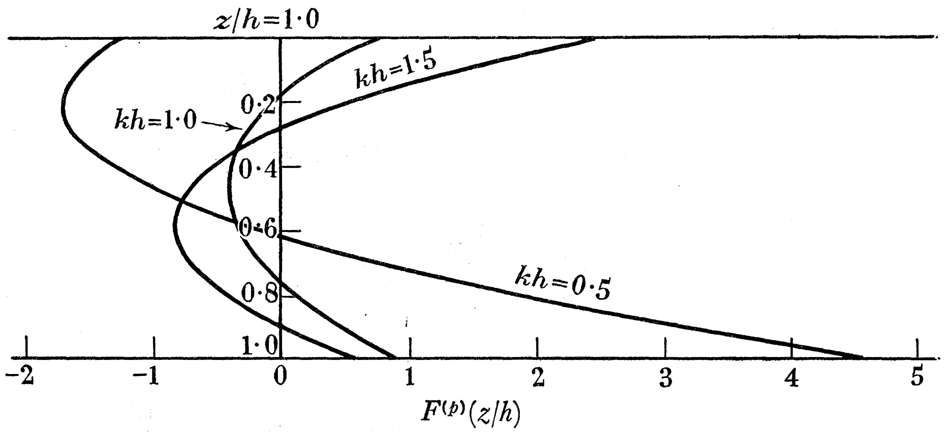

FIGURE 6. Graphs of $F^{(p)}(z/h)$ when $kh = 0{\cdot}5$, $1{\cdot}0$ and $1{\cdot}5$, representing the profile of the mass-transport velocity in the interior of the fluid in a progressive wave (conduction solution).

Equation (302) gives a parabolic velocity distribution, which is zero when $z/h = \sqrt{3}$ and has a vertical tangent at the mean free surface. However, for small values of kh the present approximation may not be good unless the wave amplitude a is very small; for the method can only be expected to remain valid if the mass-transport velocity is small compared with the orbital velocity of the particles; it will be seen that this requires $a/h \ll 1$.

For large values of kh the velocity is greatest near the free surface. When $(kh)^{-1}$ and e^{-kh} are negligible we have

$$F^{(p)}(\mu) = \tfrac{1}{2}kh(3\mu^2-4\mu+1), \tag{304}$$

and hence

$$\epsilon^2\frac{\partial\Psi}{\partial z} = \tfrac{1}{2}a^2\sigma k^2h(3z^2/h^2-4z/h+1). \tag{305}$$

This represents a parabolic velocity distribution which is zero when $z/h = 1$ and $z/h = \frac{1}{3}$, and has a vertical tangent when $z/h = \frac{2}{3}$.

(b) *The standing wave*

When $a_1 = a_2 = a$ we have from (293)

$$\epsilon^2\Psi = \frac{a^2\sigma}{2\sinh^2 kh}\sin 2kx\, Z^{(s)}(z), \tag{306}$$

$Z^{(s)}$ being given by (297). Contours of the function $\sin 2kx\, Z^{(s)}(z)$ when $kh = 1{\cdot}0$ are shown in figure 7. The circulation is in cells bounded by the vertical planes $x = \frac{1}{4}m\lambda$ (where m is any integer) and by the horizontal planes $z = 0$ and $z = h$. It may be shown that $Z^{(s)}(z)$ has only one stationary value, given by

$$2kz \tanh 2kz = 2kh \coth 2kh - 1. \tag{307}$$

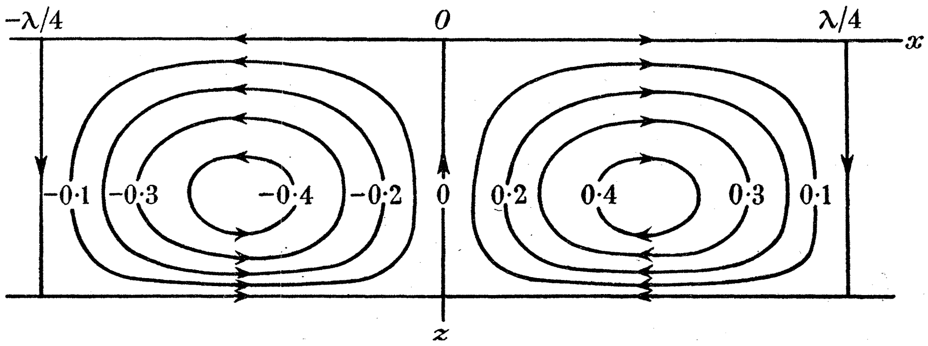

FIGURE 7. Contours of $\sin 2kx\, Z^{(s)}(z)$ when $kh = 1\cdot0$, representing the circulation of mass transport in a standing wave (conduction solution).

Hence there is only one cell in each vertical line. When kh is small we have

$$\epsilon^2\Psi' = -\tfrac{3}{4}b^2\sigma kh \sin 2kx\,(z^3/h^3 - z/h), \tag{308}$$

where b is given by (303), so that there is a point of zero velocity where

$$x = (\tfrac{1}{2}m + \tfrac{1}{4})\,\lambda, \quad z/h = \sqrt{3}. \tag{309}$$

When kh is large we find

$$\epsilon^2\Psi' = \tfrac{3}{2}a^2\sigma \sin 2kx\, \mathrm{e}^{-2kh}\, 2k(h-z)\, \mathrm{e}^{-2k(h-z)}. \tag{310}$$

This is the special case found by Rayleigh (1883).* The velocities are very small, owing to the factor e^{-2kh}. The circulation is driven by the tangential velocity near the bottom, and takes place almost entirely within a quarter of a wave-length from the bottom. There is a point of zero velocity where

$$x = (\tfrac{1}{2}m + \tfrac{1}{4})\,\lambda, \quad 2k(h-z) = 1, \quad h - z = \lambda/4\pi. \tag{311}$$

14. Motion in the interior: the convection solution

The boundary conditions for Ψ' are given, as before, by equations (286) to (289), but the field equation, from equation (84), is now

$$\nabla^2\left[\Psi' - \frac{\sigma \sinh 2k(z-h)}{4\sinh^2 kh}\,(a_1^2 - a_2^2)\right] = F(\Psi'). \tag{312}$$

(a) *The progressive wave*

In this case a solution is simply

$$\epsilon^2\Psi' = \frac{a^2\sigma}{4\sinh^2 kh}\,H(z), \tag{313}$$

* Rayleigh did not examine the motion near the free surface, or show that the mass-transport velocity gradient vanishes there. His solution is therefore incomplete, even for waves in deep water; for, a non-zero velocity gradient near the free surface would produce additional velocities near the bottom of order $a^2\sigma k\mathrm{e}^{-2kh}$, which are comparable with those produced by the tangential velocity near the bottom.

where $H(z)$ is an arbitrary function satisfying only the conditions

$$\left.\begin{aligned}\left(\frac{\partial H}{\partial z}\right)_{z=h} &= 5k,\\ \left(\frac{\partial^2 H}{\partial z^2}\right)_{z=0} &= -2k^2\sinh 2kh,\\ (H)_{z=0} &= 0,\end{aligned}\right\} \tag{314}$$

and, if the total horizontal flow is assumed to be zero,

$$(H)_{z=h} = 0; \tag{315}$$

for, equation (313) defines z as a function of Ψ, and then F can be defined by equation (312). The motion represented is a horizontal flow depending only on z. It can only be defined further if the conditions at $x = \pm\infty$ are specified.

(b) *The general case*

When neither a_1 nor a_2 vanishes, we assume, as the simplest hypothesis, that F is a linear function of Ψ:

$$F(\Psi) = \frac{C\sigma k^2}{4\sinh^2 kh}(a_1^2 - a_2^2) - r^2\Psi \tag{316}$$

where C and r are constants to be determined. The differential equation for Ψ is then

$$(\nabla^2 + r^2)\,\Psi = \frac{\sigma(a_1^2 - a_2^2)}{4\sinh^2 kh}[Ck^2 + 4k^2\sinh 2k(z-h)]. \tag{317}$$

Let

$$\epsilon^2\Psi = \frac{\sigma}{4\sinh^2 kh}\left[(a_1^2 - a_2^2)\left\{\frac{Ck^2}{r^2} + \frac{4k^2}{4k^2+r^2}\sinh 2k(z-h) + Z^{(p)\prime}(z)\right\} + 2a_1 a_2 \sin 2kx\, Z^{(s)\prime}(z)\right]. \tag{318}$$

Then $Z^{(p)\prime}$ and $Z^{(s)\prime}$ must satisfy

$$\left.\begin{aligned}\left(\frac{d^2}{dz^2} + r^2\right) Z^{(p)\prime} &= 0,\\ \left(\frac{dZ^{(p)\prime}}{dz}\right)_{z=h} &= \frac{(12k^2 + 5r^2)\,k}{4k^2 + r^2},\\ \left(\frac{d^2 Z^{(p)\prime}}{dz^2}\right)_{z=0} &= -\frac{(2k^2 + r^2)\,8k^2}{4k^2 + r^2}\sinh 2kh,\\ (Z^{(p)\prime})_{z=0} &= -\frac{Ck^2}{r^2} + \frac{4k^2}{4k^2 + r^2}\sinh 2kh,\end{aligned}\right\} \tag{319}$$

and

$$\left.\begin{aligned}\left(\frac{d^2}{dz^2} + r^2 - 4k^2\right) Z^{(s)\prime} &= 0,\\ \left(\frac{dZ^{(s)\prime}}{dz}\right)_{z=h} &= -3k, \qquad (Z^{(s)\prime})_{z=h} = 0,\\ \left(\frac{d^2 Z^{(s)\prime}}{dz^2}\right)_{z=0} &= 0, \qquad (Z^{(s)\prime})_{z=0} = 0.\end{aligned}\right\} \tag{320}$$

From the first and the last two of equations (319) it follows that

$$C = -4\sinh 2kh. \tag{321}$$

The first three equations give

$$Z^{(p)\prime} = \frac{(12k^2+5r^2)\,k}{(4k^2+r^2)\,r}\,\frac{\sin rz}{\cos rh} + \frac{(2k^2+r^2)\,8k^2}{(4k^2+r^2)\,r^2}\sinh 2kh\,\frac{\cos r(z-h)}{\cos rh}. \tag{322}$$

Equations (320) possess a solution

$$Z^{(s)\prime} = -3k\,\frac{\sin\,(r^2-4k^2)^{\frac{1}{2}}\,z}{(r^2-4k^2)^{\frac{1}{2}}\cos\,(r^2-4k^2)^{\frac{1}{2}}\,h}, \tag{323}$$

provided that

$$r^2 = 4k^2 + m^2\pi^2/h^2, \tag{324}$$

where m is a positive integer. In this case (323) may also be written

$$Z^{(s)\prime} = (-1)^{m+1}\,\frac{3kh}{m\pi}\sin\,(m\pi z/h). \tag{325}$$

Once m is chosen, both $Z^{(s)\prime}$ and $Z^{(p)\prime}$ are completely defined. There is an infinite number of solutions, each corresponding to a different integer m, but solutions corresponding to different values of m are not of course superposable. Now when $z = h$ we have

$$\epsilon^2\Psi = \frac{\sigma(a_1^2-a_2^2)}{4\sinh^2 kh}\left[-\frac{4k^2}{r^2}\sinh 2kh + \frac{(12k^2+5r^2)\,k}{(4k^2+r^2)\,r}\tan rh + \frac{(2k^2+r^2)\,8k^2}{(4k^2+r^2)\,r}\sinh 2kh\sec rh\right]. \tag{326}$$

It may be shown that the expression in square brackets cannot vanish when $kh > 0$. Hence the present solution does not represent a motion having zero total horizontal flow, except when $a_1^2 = a_2^2$ (the case of the standing wave).

(c) *The standing wave*

When $a_1 = a_2 = a$ we have from (318) and (315)

$$\epsilon^2\Psi = (-1)^{m+1}\,\frac{3}{2}\,\frac{\sigma kha^2}{m\pi\sinh^2 kh}\sin 2kx\sin\,(m\pi z/h). \tag{327}$$

This represents a circulation in cells similar to those in the conduction solution (§ 13), except that in each vertical line there are now m cells instead of only one as formerly. The vertical boundaries of the cells are the planes $x = \frac{1}{4}m'\lambda$ (where m' is any integer), and the horizontal boundaries are the planes $z = m''h/m$ $(0 \leqslant m'' \leqslant m)$. The circulations in adjacent cells are in opposite senses; those in the lowest cells are in the same sense as the circulations in the corresponding cells in the conduction solution.

The results of the present section may be summarized by saying that for the progressive wave the convection solution is arbitrary, for the standing wave there is an infinity of possible solutions, and in the general case of two opposite waves of unequal amplitude there exists an infinity of homogeneous solutions of the present type; these, however, represent motions with non-zero total horizontal flow.

15. Discussion

The conduction and convection solutions for the first-order motion which is represented by (244) are exact, to the present degree of approximation. However, owing to the dissipation of energy by viscosity, equation (244) itself is only approximate; for the motion cannot be exactly periodic in both space and time. The assumption usually made (see Basset 1888; Hough 1896) is that the motion is periodic in space and has a small decrement in time. But since one of our fundamental assumptions is that the motion is periodic in time, we must here suppose that the motion is attenuated in a horizontal direction. For a progressive wave in which the energy is propagated in the direction of x increasing, there will be an exponential decrease with x; instead of a 'standing wave' we may consider the sum of two progressive waves attenuated in opposite directions.

Now the energy dissipation E per unit volume is proportional to $\rho\nu$ times (velocity gradient)^2. Thus in the interior of the fluid, and in the boundary layer at the free surface, E is only of order $\rho\nu a^2\sigma^2k^2$, or $\rho g a^2\sigma k(\delta/l)^2$, where l is the wave-length. But in the boundary layer at the bottom the velocity gradients are of order $a\sigma/\delta$, and hence the energy dissipation is of order $\rho g a^2\sigma k$ per unit volume, or $\rho g a^2\sigma(\delta/l)$ per unit area of the bottom. Thus most of the energy dissipation takes place in the boundary layer at the bottom, provided the depth is not too great. But the transfer of energy horizontally can be shown to be almost independent of the viscosity, so that the proportional rate of attenuation horizontally is of order δ/l per unit wave-length at most. This is of the same order as quantities already neglected.

The conduction solution for the progressive wave given in § 13, which is independent of the horizontal co-ordinate x, satisfies also the convection equations; for the stream-lines are parallel, and the vorticity along each is constant. It might therefore be supposed that the solution is valid for all values of a^2/δ^2. However, if the horizontal attenuation of the waves is taken into account there must be a small vertical component of velocity, and the conduction terms no longer vanish identically. It then becomes difficult to find a convection solution. The conduction solution, on the other hand, can easily be modified to take account of the attenuation. Since the vertical velocities are small it is possible that, for the progressive wave, the range of validity of the conduction solution (for which it was specified that $a^2/\delta^2 \ll 1$) is slightly greater than that assumed. However, in the case of the standing wave, where the convection terms do not vanish identically, the condition $a^2/\delta^2 \ll 1$ cannot be relaxed.

Let us now consider the possible sequence of events, supposing that the motion is started from a state of rest. For definiteness suppose that waves are generated in a rectangular tank of length L, width D, and depth h (where L is large compared with a wave-length) by an oscillating plunger or paddle at one end of the tank. If a progressive wave is considered, the waves may be supposed to be dissipated by a 'beach' or wave absorber at the far end of the tank, or they may be partially or wholly reflected by a suitable obstacle placed in the tank; if they are wholly reflected a standing wave is formed.

Observation has shown (see Cooper & Longuet-Higgins 1950) that the wave energy travels down the tank with approximately the theoretical group velocity $g/2\sigma$, and that soon after the passage of the 'wave front' the first-order motion is well established. The mass-transport distribution in the boundary layers can be expected to be set up almost immediately, for it depends, as was shown in part II, only on the first-order motion and on the local boundary conditions. There may be some departures from the theoretical velocity distribution owing to the presence of large velocity gradients just beyond the boundary layer, for these might not at first be small compared with the velocity gradients in the boundary layer itself, as was assumed. But after a few cycles the velocity gradients just beyond the boundary layer can be expected to be smoothed out by the viscosity.

In the interior of the fluid the motion will at first be irrotational, since no vorticity can be generated there. The mass-transport distribution should therefore be as described by Stokes (1847). Subsequently the nature of the motion will depend upon the ratio a^2/δ^2. If $a^2/\delta^2 \ll 1$, that is, for very small waves indeed, the motion would be as described by the conduction solutions of § 13 (except possibly near the vertical sides of the tank, where the motion has not yet been considered). In order that the solution should be valid it must be supposed that the width D of the tank is great compared with the depth h of water. By analogy with the diffusion of heat, the time taken for the vorticity to diffuse into the interior and for a steady state to be reached will be of the order of h^2/ν.

In nearly all practical cases, however, we shall have $a^2/\delta^2 \gg 1$, so that, if a steady state exists, it is given by the convection solution of § 14. For the progressive wave, this solution is arbitrary, or rather it depends on the boundary conditions imposed at $x = \pm\infty$. In practice, therefore, we may expect that the motion will depend upon the special conditions at the wave maker and the wave absorber respectively; vorticity will be generated at these points and will be diffused horizontally along the stream-lines. The time taken for the whole interior of the tank to be affected in this way is of the order of $L/(a^2\sigma k)$. In the meantime, some vorticity will be diffused inwards from the bottom, from the free surface and from the vertical sides by viscous conduction. The width affected in this manner is of the order of $(L\nu/a^2\sigma k)^{\frac{1}{2}}$ (this quantity is assumed to be small compared with h or D). However, it is by no means certain that a steady state will exist which is compatible with the boundary conditions at both the wave maker and at the wave absorber, or that, if it exists, it is stable. The situation is even less predictable when one considers a partially reflected wave, for which no convection solution satisfying the condition of zero total transport has been found, or the standing wave, for which there is an infinity of such solutions.

16. Comparison with observation

It appears from the preceding discussion that the theory can best be compared with observation, first, in the boundary layers, where the motion is well-determined irrespective of the ratio a^2/δ^2, and, secondly, in the interior of the fluid before vorticity has had time to be diffused inwards; the motion should then be described by Stokes's irrotational theory. Not many quantitative determinations of mass-transport velocity have been made under controlled conditions, but the chief observations will now be discussed.

Caligny (1878)

The earliest quantitative observations seem to be due to Bertin & Caligny (Caligny 1878). These authors used a tank of length 29·7 m, depth 47 cm and width 50 cm; the depth

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of water was 36 cm. Waves were generated at one end by a steam-driven plunger, travelled down to the far end and were dissipated on a sloping plane 'beach'. The movement of particles of resin suspended in the fluid was observed through glass windows in a side wall of the tank. Caligny gives the following values of the mean horizontal velocity for waves of period 1 s, wave-length 130 cm and height 6 cm:

distance above bottom (cm)	0	5	9	15	23	27	36
mean velocity (cm/s)	0·4	0·0	−0·3	−0·5	0·0	0·3	0·5

This shows a forward velocity both near the bottom and near the free surface, with a negative velocity between. Assuming $\sigma = 2\pi\,\mathrm{s}^{-1}$, $k = 2\pi/130\,\mathrm{cm}^{-1}$, $h = 36$ cm and $a = 3$ cm we find that the theoretical velocity just in the interior of the fluid, according to equation (255), is 0·45 cm/s, in good agreement with the observation at the lowest level. The velocity gradient near the free surface was not recorded; the theoretical value given by equation (276) is $-0{\cdot}56\,\mathrm{s}^{-1}$, compared with a mean value of $-0{\cdot}02\,\mathrm{s}^{-1}$ between the two observations nearest the upper surface. Caligny, however, mentions that the observations at the uppermost levels were rather scattered. This might either be because the motion was not steady, or because the velocity gradient was so large that the velocity depended critically on the depth of the particle of resin below the free surface.

In some previous but less precise experiments (1861) Caligny had observed a backward movement of grains of sand and resin on the bottom. But this movement diminished rapidly with distance from the wave maker and seems to have been due to the fact that locally the waves were not progressive.

Mitchim (1939)

A systematic experimental study of deep-water waves was made by Mitchim (1939) using a tank 60 ft. long, 1 ft. wide and 3 ft. deep. The depth of water was 2·5 ft., and the wave-lengths investigated were from 2 to 5 ft., or less than twice the depth of water; thus the waves were, effectively, in deep water. The motion was generated at one end of the tank by a wooden flap hinged on the bottom, and was dissipated at the far end on a sloping plane beach. The mass-transport velocities below the free surface were measured by photographing the tracks of white liquid particles suspended in the fluid; the velocities at the surface were measured by observing the progress of a small wooden cylinder $\frac{1}{16}$ in. in diameter.

The surface velocities agreed fairly well with the irrotational theory, being mostly within 20 %. The velocities in the interior were in qualitative agreement with the irrotational theory, being forward near the surface and backward at the lower levels; but the scatter of the observations, even on the same run, was such that it seems unlikely that a steady state had been reached. No observations very near the bottom are reported.

The United States Beach Erosion Board (1941)

Some mass-transport observations are included in an experimental study of surface waves by the United States Beach Erosion Board (1941). The wave-lengths λ used were between 3·5 and 12·2 ft., and the depth of water was between 1 and 3 ft. There were no observations near the bottom, nor is it stated for how long the waves had been running at the time of the observation. In deep water ($h > \frac{1}{2}\lambda$) there was reasonable agreement with

Stokes's theory in the upper half of the fluid (though no observations within 2 in. of the surface are given); in shallow water ($h < \frac{1}{2}\lambda$) the agreement with Stokes's theory was poor and the observations show considerable scatter; it seems unlikely that a steady state had been reached.

Bagnold (1947)

In the course of a study of the movement of sand by water waves, Bagnold (1947) also made observations of the motion of the water particles themselves. His apparatus consisted of a glass-sided channel 11 m long, 30 cm wide and 30 cm deep, opening at one end into a slightly deeper channel 3 m long. A paddle hinged at the bottom of the deeper channel generated waves which travelled down the channel and were dissipated on a beach of pebbles or sand. To observe the mass transport, grains of dye impregnated with fluorescein were inserted into the water; these fell to the bottom, leaving a vertical streak which then gradually deformed, giving a direct picture of the velocity profile.

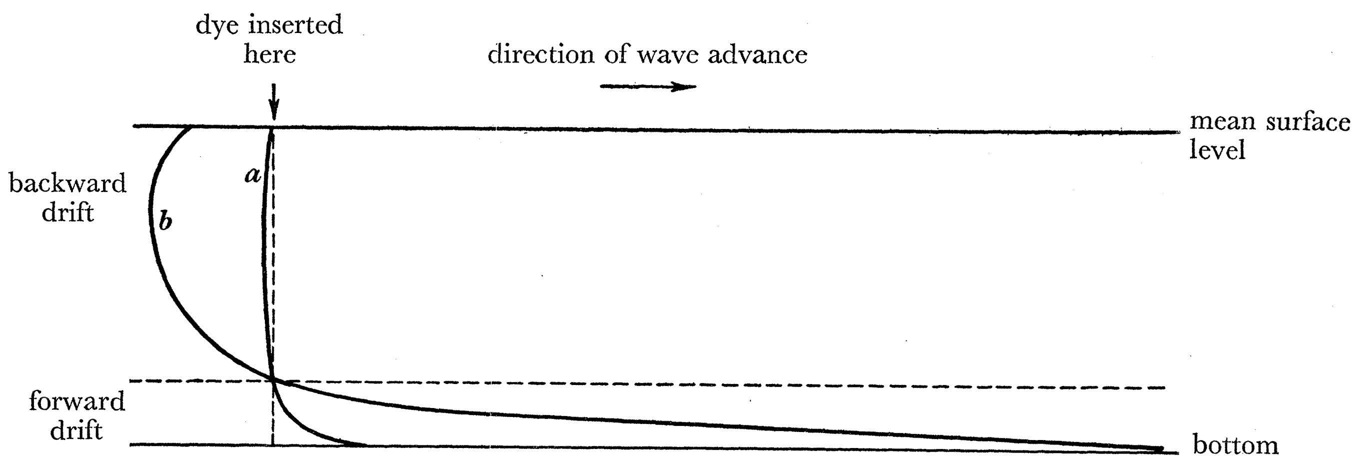

FIGURE 8 (after Bagnold 1947). Successive positions of the dye streak, indicating the profile of the mass-transport velocity (*a*) after one wave (*b*) after 10 waves.

Bagnold's first observations were made with a sandless bed, the bottom being of painted wood. His sketch of a typical velocity profile is reproduced in figure 8. It shows a strong forward drift near the bottom, and a weaker backward drift at higher levels. The uppermost part of the profile was unsteady; but in all cases there was a forward bend at the top of the curve.

The velocity of the foremost tip of the dye was observed; Bagnold's two series of observations are tabulated in the final column of table 1 (*a*) and (*b*). The parameters used by him to define the motion were the period $2\pi/\sigma$, the wave height $2a$ and the height of the wave troughs above the bottom ($h-a$ in the present notation). For each observation the non-dimensional parameter $\sigma^2 h/g$ has been calculated, and kh found from equation (245). In the fourth column of table 1 is given the theoretical maximum velocity in the boundary layer, calculated from equation (259).

The agreement between the last two columns of table 1 (*a*) is within 15 %, which is satisfactory considering the errors probably involved in the observations. In table 1 (*b*) there is good agreement at the two ends of the range of observation, but some discrepancy for intermediate values. No explanation of the 'kink' in the experimental curve has been found.

On reaching the point at which the waves broke, the dye was observed to rise vertically from the bottom and to become dispersed in the upper layers, which drifted slowly away from the shore. From the velocities in table 1 we should expect that the motion, if controlled by convection, would be established in a few minutes. After starting the paddle, a few minutes were always allowed for the motion to settle down; afterwards the velocity profile remained the same shape indefinitely. However, the initial drift profile could not be observed very well owing a to 'seiche' which was set up in the tank when the motion was started.

TABLE 1. COMPARISON OF THE OBSERVED AND THEORETICAL MASS-TRANSPORT VELOCITIES NEAR THE BOTTOM IN A PROGRESSIVE WAVE

τ (s)	$\sigma^2 h/g$	kh	$U_{\text{max.}}$ (cm/s)	$U_{\text{obs.}}$ (cm/s)
		(*a*) $a = 3{\cdot}0$ cm, $h = 16{\cdot}0$ cm		
0·78	1·05	1·24	3·1	3·0
0·88	0·83	1·06	3·6	3·2
0·95	0·71	0·96	3·8	3·6
1·08	0·55	0·82	4·4	3·8
1·32	0·37	0·65	4·9	4·1
1·58	0·26	0·53	5·3	4·6
		(*b*) $a = 1{\cdot}55$ cm, $h = 14{\cdot}5$ cm		
0·59	1·67	1·77	0·5	0·5
0·78	0·96	1·17	1·0	1·4
0·98	0·61	0·87	1·3	2·2
1·09	0·49	0·76	1·4	2·0
1·30	0·35	0·63	1·5	1·8
1·57	0·24	0·51	1·6	1·6

Similar observations to those of Bagnold, but on an inclined wooden ramp, were made by King (1948). In this case the forward movement was found both near the bottom and near the free surface, with backward movement between.

Conclusions

The strong forward velocities near the bottom, which were observed by Bagnold and by Bertin & Caligny, are accounted for quantitatively by the present theory. In a progressive wave we may expect a forward bending of the velocity profile near the free surface—twice that predicted by the irrotational theory—but no careful observations are yet available. In the standing wave there should be a circulation in the bottom boundary layer in cells of width one-quarter of a wave-length. Although there are some indications from the motion of sand particles that this may be so, there is as yet no direct experimental verification.

The observations of mass transport in the interior of the fluid may be divided into two classes: those in deep water and those in shallow water. In deep water the observations seem to be not greatly different from those predicted by the irrotational theory—as one would expect if the waves had not been running for long. In shallow water the observations appear to be very scattered; it is uncertain whether, in any of the observations quoted, a steady state had been reached.

Further experiments are desirable to determine the range of validity of the boundary-layer theory for progressive waves, to verify the results predicted for standing waves and to determine whether the motion in the interior is stable.

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